Misspecified Politics and the Recurrence of Populism

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Abstract: We develop a model of political competition between types that differ in their subjective model of the data generating process for a common outcome. We show that political competition does not weed out misspecified models which are simpler as they ignore some relevant policy variables. Specifically, periods in which those with a correctly specified and more complex model govern increase the specification error of the simpler world view, leading the latter to underrate the effectiveness of complex policies and overestimate the positive impact of a few extreme policy actions. Periods in which simple types implement their narrow world view result in subpar outcomes and a weakening of their omitted variable bias. Policy cycles arise, where each type's tenure in power sows the seeds of its eventual electoral defeat.

"Democracy is complex, populism is simple" (R. Dahrendorf)

I Introduction

Individuals differ not merely in their economic interests and preferences, but also in their fundamental understanding of the data generating process that underlies observed outcomes. Consequently, because they consider the same historical data through the prism of different models, fully rational and otherwise similar actors can have persistent differences of opinion, as witnessed by the endurance of academic debates in areas as diverse as macroeconomics and physics. In politics, such differences in model specification translate into differences in realized policy decisions when different groups are in power. The consequent interplay between beliefs and policy can generate systematic correlations between observed data that sustain differing beliefs and biases.

This paper considers political competition between types that share the same interests and preferences over common outcomes but differ in their subjective models of the causes

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of these outcomes. Because of the infinite number of potential regressors and finite number of observations, all actors must restrict the set of policies they consider relevant, i.e. may have non-zero effects on the common outcome. Our principal aim is to investigate whether political competition weeds out actors with simplistic misspecified models, and what are the eventual long term beliefs of the actors in this polity.

Specifically, we consider the following dynamic model. Output is a simple linear function of a set of policy variables as well as a random shock. Everyone in the polity is interested in maximizing this output (subject to a resource constraint), but individuals differ in their subjective models of the relation between policy variables and output. A "complex" type has a correctly specified model in that it knows that all these variables affect the outcome, and a "simple" type has a misspecified model and considers a smaller set of relevant variables. For example, while a complex type may consider crime as best treated with a range of policies, a simple type come to view crime as stemming from a single cause, e.g. policing. Both types start with a prior and overtime learn about the parameters determining the magnitude of the effect of each policy variable on the actual outcome.

We assume that political competition takes a simple form so that the type that wins is the one that has a higher intensity of preferences (that is, the type that is more keen on winning the election rather than letting the other side win). This type chooses her ideal configuration of policies which are then implemented with small "bureaucratic" noise. At every period output is observed and both types use OLS to update their beliefs. Note that over time, observations are not iid as learning and hence current policies depends on previous shocks.

In our key result in Section III we show that the dynamic process converges to a unique steady state. This steady state is characterized by two important features. First, the complex type, which has the correct model, is unable to permanently defeat and remove from power those with misspecified simple beliefs. Equilibrium is therefore characterized by power sharing between the two types (and hence equal intensity of preferences). When the complex govern and implement their broad policy agenda this increases the omitted variable bias of the simple, as they attribute the successful outcomes of the full range of complex policies to moderate actions taken on a few dimensions. This increases the simple's assessment of the likely effectiveness of a more decisive narrow policy and mobilizes them in support of political candidates who will implement it. However, when the simple govern they produce systematically inferior results, as their extreme actions are revealed to be less effective than anticipated. This reduces the intensity of both their desired policy and political activism, thereby allowing complex types to regain power. Thus, we find that the economy suffers from inevitable political cycles and the recurrence of inefficient policies.

The second feature of the steady state is the connection between simple world views and extremism. We show that the beliefs of the simple type converge to a multiple, larger than one, of the corresponding beliefs of the complex. As a result, when in power, the simple implement a narrowed and exaggerated version of complex policies. Intuitively, when policies of the two types are not collinear, there is enough variation in the data so that the simple type approximate the expected average outcomes of polices both when they are in power and when the complex are in power. However, as we show, this induces the simple type to become more eager to win the election and thus contradicts equilibrium power sharing. In short, the simple type cannot learn too much in equilibrium, leading to inflated beliefs on all the policies it considers.

In the unique equilibrium we find there are perpetual transitions of power between the complex and the simple types, who implement extreme and ineffective policies. In this sense, our model may shed some light on the recurrence of political populism. The amorphous concept of "populism" has perhaps as many definitions as authors. Simple world views, while not the only feature of populism, are an important aspect of such movements. For example, many recent theories focus on the anti-establishment rhetoric of populism,[2] which represents the "will of the ordinary people". Almost by definition, the will of the people is simple; it has to be a common ground of many. An additional frequent theme is that the policies of populist politicians are extreme, misguided and harmful to the very groups that support them (e.g., Dornbusch and Edwards 1991). Our framework provides a motivation for the recurrence of large policy deviations with subpar outcomes that are supported by rational voters.

Our theoretical contribution is to establish convergence in a learning environment with a misspecified model. Convergence of beliefs in such environments is not guaranteed, and is especially problematic with multidimensional state spaces (Heidhues, Kőszegi & Strack 2018, Bohren & Hauser 2019, Esponda, Pouzo & Yamamoto 2019, and Frick, Iijima & Ishii 2020). Our paper provides an example of how convergence can be proven in a model with multiple agents, a multidimensional state space and continuous actions. Specifically, we use noise in the implementation of policies to establish convergence in an OLS framework.

In Section IV we also consider a static notion of equilibrium in the spirit of Berk-Nash equilibria.[3] This allows us to study more general Bayesian environments. We show that an equilibrium analogous to the unique equilibrium above, with political cycles and extremism, is a Berk-Nash equilibrium. We also show that any Berk-Nash equilibrium of our model

[2] See Mudde and Kaltwesser (2017).

[3] Esponda and Pouzo (2016, 2018) explore the implications of model misspecification by suggesting the Berk-Nash (BN) equilibrium. In a BN equilibrium agents play optimally with respect to the model that is the best fit, i.e. the model that minimizes relative entropy with respect to the true distribution of outcomes under the equilibrium strategy profile.

involves inefficient policies.

Interest in learning with misspecified models dates back at least to Arrow & Green (1973), with examples including Bray (1982), Nyarko (1991), Esponda (2008) and, most recently, Esponda and Pouzo (2016) and Molavi (2019). Several recent papers feature interactions between competing subjective models that share features of our framework. Mailath and Samuelson (2019) consider individuals with heterogenous models who exchange beliefs sequentially once they receive a one-off (private) data and characterize conditions under which beliefs converge. Eliaz and Spiegler (2019) present a static model of political competition based upon competing narratives that draw voters' attention to different causal variables and mechanisms. They focus on a static equilibrium and on the possibility of "false positive" variables (which are not necessarily policy variables). Montiel Olea et al (2017), with auctions as a motivation, consider competition between agents that use simple or complex models to explain a given set of exogenous data and find that simpler agents have greater confidence in their estimates in smaller data sets and less confidence asymptotically. In our framework the endogenous data produced by actors with different specifications generates persistent biases and differences in beliefs that asymptotically keep both types politically competitive.

Our paper builds on a literature of political-economy models of sub-optimal populist policies. Acemoglu et al (2013) model left-wing populist policies that are both harmful to elites and not in the interests of the majority poor as arising from the need for politicians to signal that they are not influenced by rich right-wing interests. Di Tella and Rotemberg (2016) analyze populism in a behavioural model in which voters are betrayal averse and may prefer incompetent leaders so as to minimize the chance of suffering from betrayal. Guiso et al (2017) define a populist party as one that champions short-term redistributive policies while discounting claims regarding long-term costs as representing elite interests. Bernhardt et al (2019) show how office seeking-demagogues who cater to voters' short term desires compete successfully with far-sighted representatives who guard the long-run interests of voters. Morelli et al (2020) show how in a world with information costs incompetent politicians who simplistically commit to fixed policies can be successful. Our framework expands this literature by linking the pursuit of sub-optimal policy to the bias created by a misspecified interpretation of the outcomes of periods of optimal rule.

The paper proceeds as follows: Section II presents our basic framework, wherein voters differ in their beliefs regarding the possible determinants of common outcomes. Section III establishes the convergence to the unique steady state and discusses its implications. In Section IV we discuss several extensions and modelling assumptions. In particular we discuss the relation between the unique equilibrium we characterize and the Berk-Nash equilibria of this model. An appendix contains all proofs not in the text.

II The Model

The Economic Environment: We consider a common outcome $y \in R$ whose realization at time t is governed by the data generating process:

$$(II.1) \quad y_t = (\mathbf{x}_t + \mathbf{n}_t)'\boldsymbol{\beta} + \varepsilon_t$$

where $\mathbf{x}_t$ and $\boldsymbol{\beta}$ are vectors of k policy actions in $\mathbb{R}^k$ and associated parameters, and $\varepsilon_t \in \mathbb{R}$, a mean zero iid normally distributed random shock.[4][5] We assume that all elements of $\boldsymbol{\beta}$ are non zero. The term $\mathbf{n}_t \in \mathbb{R}^k$ is a $k-$vector of policy noise which could be thought of as small policy implementation shocks. The components of noise $\mathbf{n}_t$ are iid with zero mean and diagonal covariance matrix $\sigma_n^2 \mathbf{I}_k$, and are independent of both the policy vector $\mathbf{x}_t$ and the shock to outcomes ε_t. We add noise to all relevant k policies, but alternatively we could add noise to only the set of policies that are implemented at each period and the results would be the same.

Although y is described as a single outcome, one can equally think of it as a preference weighted average of multiple outcomes that are influenced by $\mathbf{x}_t$.[6] Below, we use bold letters to denote vectors and when it does not lead to confusion, often drop the subscript t, writing $\mathbf{x}$, $y, \mathbf{n}$ and ε.

Subjective Models: We assume that citizens are divided into two "types" based upon their subjective model about which of the unknown parameters in $\boldsymbol{\beta}$ can potentially be non-zero. We shall focus our analysis on the case where "complex" types (C) that believe all elements of $\boldsymbol{\beta}$ might be non-zero compete politically with "simple" types (S) whose model is misspecified, in that the policies they think are relevant exclude some of the non-zero elements of $\boldsymbol{\beta}$. We assume that it is common knowledge that ε is normally distributed.

We use the subscript i to distinguish between the full k x 1 vectors of effective policies and parameters ($\mathbf{x}$ and $\boldsymbol{\beta}$) and the $k_i \leq k$ sub-elements of these that type $i \in \{S, C\}$ thinks are potentially relevant ($\mathbf{x}_i$ and $\boldsymbol{\beta}_i$). Specifically, $k_s < k_c = k$. In addition, we denote by $\mathbf{x}_{ij}$ the vector of policies that i finds relevant and are implemented when j is in power. While

[4] We can generalise our results to allow for a constant term in the output function under some additional assumptions.

[5] In Section IV where we study Berk Nash equilibria of our model we will consider more general distributions of the shock, $f(\varepsilon)$.

[6] If utility is a weighted average of i components each with $y_{it} = (\mathbf{x}_t + \mathbf{n}_t)'\boldsymbol{\beta}_i + \varepsilon_{it}$, then the outcome, parameters and error term in $II.1$ are simply the weighted average of those components.

the subjective model of type $i \in \{S, C\}$ is fixed, the beliefs of type $i \in \{S, C\}$ about the magnitude of the elements in $\boldsymbol{\beta}_i$ will evolve over time according to OLS estimation.

Below we will assume linear utility; together with the linear formulation of y, this implies that only mean beliefs will matter, and we will henceforth denote the vector of mean belief at period t by $\bar{\boldsymbol{\beta}}_s$ and $\bar{\boldsymbol{\beta}}_c$ respectively.

We use $\mathbf{H}_t = \mathbf{X}_t + \mathbf{N}_t$ to denote the t x k history of desired policy and iid noise. Each type will use the associated t x k_i columns $\mathbf{H}_i$ of $\mathbf{H}$ in a regression model to derive their mean belief $\bar{\boldsymbol{\beta}}_i$. We assume that prior beliefs are normally distributed. As our results are in any case asymptotic, normal beliefs of this sort can be justified by the observation of a long pre-history of policy, as under fairly general conditions the likelihood function determines the shape of the posterior (Zellner 1971).[7] As the error ε is independent of contemporaneous policy, period by period updating then leads to mean posterior beliefs (during the period of analysis) in the form of the OLS estimates:

$$(II.2)\ \bar{\boldsymbol{\beta}}_{it} = (\mathbf{H}_{it}'\mathbf{H}_{it})^{-1}\mathbf{H}_{it}y_t.$$

In Section IV we consider a more general model where each group can also believe that non-relevant policies affect y, and so initially S may consider more policies than C. We show that as long as S considers a subset of the *relevant* policies that C considers, they end up with a simpler model of the world.

Preferences and optimal policies: We model utility with the minimal structure that allows for a tractable presentation. Specifically, we assume the utility citizens derive from the common outcome is linear:

$$(II.3)\quad U_t(y_t) = y_t,$$

and that the choice of policies is subject to a budget constraint, and so $\mathbf{x}_t'\mathbf{x}_t \leq R$, where R is some bounded, exogenously-given, resources. The constraint is formulated so that it allows us not to worry about the signs of the elements of $\boldsymbol{\beta}$ or $\mathbf{x}$.

Given the above, it readily follows that:

Lemma 1: *At any period, given some mean belief $\bar{\boldsymbol{\beta}}_i$ for type $i \in \{S, C\}$, the optimal myopic policy solves*

$$(II.4)\qquad \max_{\mathbf{x} \in \mathbb{R}^{k_i}} \bar{\boldsymbol{\beta}}_i'\mathbf{x} + \lambda(R - \mathbf{x}'\mathbf{x})$$

[7] Specifically, consider prior beliefs for each type across the policies they believe are relevant are normally distributed with mean $\bar{\boldsymbol{\beta}}_{i0}$ and joint covariance matrix $\sigma_{i0}^2\mathbf{V}_{i0}^{-1}$, while the prior probability density function on σ_{i0}^2 is inverted gamma. We then define the pre-history such that $\mathbf{V}_{i0} = \mathbf{H}_{i0}'\mathbf{H}_{i0}$ and $\bar{\boldsymbol{\beta}}_{i0} = (\mathbf{H}_{i0}'\mathbf{H}_{i0})^{-1}\mathbf{H}_{i0}'\mathbf{y}_0$.

resulting in

$$(II.5) \qquad \lambda = \frac{1}{2}\sqrt{\frac{\bar{\boldsymbol{\beta}}_i'\bar{\boldsymbol{\beta}}_i}{R}},\ \mathbf{x}_i^* = \bar{\boldsymbol{\beta}}_i'\sqrt{\frac{R}{\bar{\boldsymbol{\beta}}_i'\bar{\boldsymbol{\beta}}_i}} \Rightarrow \bar{y}[\mathbf{x}_i^*, \bar{\boldsymbol{\beta}}_i] \equiv \bar{\boldsymbol{\beta}}_{\mathbf{i}}'\mathbf{x}_i^* = \sqrt{\bar{\boldsymbol{\beta}}_i'\bar{\boldsymbol{\beta}}_i}\sqrt{R}$$

While the solution to the Lagrangian problem is straightforward, we note here that given the constraint R, types which have more extreme parameter estimates, as measured by $\bar{\boldsymbol{\beta}}_i'\bar{\boldsymbol{\beta}}_i$, believe they know how to pursue more effective policies, as measured in $\bar{y}[\mathbf{x}_i^*, \bar{\boldsymbol{\beta}}_i]$, and consequently feel more constrained by the resource limitation R, as measured by λ. We will show that this will feed into their relatively higher intensity of preferences to win election and choose policies.

In each period political competition will determine which type will choose current period policies. We now describe the model of political competition.

The political competition: We first define the notion of intensity of preferences. Let

$$(II.6) \qquad I_i = E_i[\bar{y}(\mathbf{x}_i^*, \bar{\boldsymbol{\beta}}_i) - \bar{y}(\mathbf{x}_j^*, \bar{\boldsymbol{\beta}}_i)],$$

where E_i denotes the expectation based upon the beliefs on $\bar{\boldsymbol{\beta}}_i$ of each type and $\bar{y}(\mathbf{x}_j^*, \bar{\boldsymbol{\beta}}_i)$ is type $i's$ expected outcome when type j chooses their optimal policy. The intensity of preferences of type i is therefore the loss this type incurs from type $j's$ ideal policy compared to her own ideal policy, given her subjective model. I_i does not necessarily equal $-I_j$ as beliefs differ across the two types. We then have:

$$(II.7) \qquad \begin{aligned} I_s &= \bar{\boldsymbol{\beta}}_s'\mathbf{x}_s^* - \bar{\boldsymbol{\beta}}_s'\mathbf{x}_{sc}^*, \\ I_c &= \bar{\boldsymbol{\beta}}_c'\mathbf{x}_c^* - \bar{\boldsymbol{\beta}}_c'\mathbf{x}_{cs}^*, \end{aligned}$$

We assume that at any period t, the type that has higher intensity of preferences wins the election (and implements her ideal policy). Below we construct a political competition model which rationalizes this assumption.

Assume that the polity consists of two equally sized groups, simple and complex, each a continuum. Each group is represented by a "citizen-candidate" that runs in the election and if elected, implements the type's ideal policy.[8] Voting is costly, but citizens vote because they believe that with some (exogenous) probability p their vote will be pivotal.[9] Consequently, a voter l of type i will vote (for their own representative) if the expected gain from the

[8] Given how we model voting decisions, it is easy to see that the presence of such candidates, offering voters of each type their ideal policy, will drive out all other policy platforms.

[9] For simplicity we are not modelling strategic voting, i.e., p is not determined endogenously in the model. The parameter p could be interpreted as the perception of voters about the probability they are pivotal in the election.

implementation of type $i's$ optimal policies relative to those of type j exceeds voter $l's$ cost of voting, c_l, i.e.:

$$(II.8) \quad pI_i > c_l$$

We assume that c_l is iid drawn from a distribution of voting costs $G(c)$ and that the cost distribution is the same for both groups. Thus, the vote share that candidates of each type garner will be an increasing function of the intensity of their type. Consequently, the election is won by the candidate representing the type with the greatest preference intensity. The results below can be generalized to allow for unequal group sizes and different distributions. For example, the case of unequal groups implies the smaller group will require a certain margin of voting preference intensity to motivate its base enough to win an election.

Before defining our equilibrium notion, we now characterize voters' intensity of preferences:

Lemma 2: *Intensity of preferences for type i is an increasing function of $\bar{\boldsymbol{\beta}}_i'\bar{\boldsymbol{\beta}}_i$, hence:*

$$(II.9) \qquad I_i > I_j \text{ iff } \bar{\boldsymbol{\beta}}_i'\bar{\boldsymbol{\beta}}_i > \bar{\boldsymbol{\beta}}_j'\bar{\boldsymbol{\beta}}_j$$

To see how this arises, note that the gain in expected utility for a voter from pursuing an optimal policy $\mathbf{x}^*$ versus an alternative policy in which a $k \times 1$ vector $\boldsymbol{\delta}$ is added to $\mathbf{x}^*$, denoted by $\mathbf{x}^*_{+\boldsymbol{\delta}}$, that satisfies the same resource constraint is given by:

$$(II.10) \qquad \bar{y}[\mathbf{x}^*, \bar{\boldsymbol{\beta}}] - \bar{y}[\mathbf{x}^*_{+\boldsymbol{\delta}}, \bar{\boldsymbol{\beta}}] = -\boldsymbol{\delta}'\bar{\boldsymbol{\beta}}$$

Substituting using optimal policies and the fact that $-\boldsymbol{\delta}'\mathbf{x}^* = \frac{1}{2}\boldsymbol{\delta}'\boldsymbol{\delta}$, as both $\mathbf{x}^{*\prime}\mathbf{x}^*$ and $(\mathbf{x}^*_{+\boldsymbol{\delta}})'(\mathbf{x}^*_{+\boldsymbol{\delta}})$ equal R, we see that:

$$(II.11) \qquad \bar{y}[\mathbf{x}^*, \bar{\boldsymbol{\beta}}] - \bar{y}[\mathbf{x}^*_{+\boldsymbol{\delta}}, \bar{\boldsymbol{\beta}}] = \sqrt{\frac{\bar{\boldsymbol{\beta}}'\bar{\boldsymbol{\beta}}}{R}} \frac{\boldsymbol{\delta}'\boldsymbol{\delta}}{2}$$

As a result, individuals with more extreme parameter estimates feel the resource constraint more keenly and hence lose more from a sub-optimal movement $\boldsymbol{\delta}$ away from their constrained choice. Hence the dynamic change of power in our model will be determined by the relative magnitude of beliefs of the two types.

Dynamics: We consider the following dynamic process:

1. In any period t, the winning type $i \in \{S, C\}$, chooses her ideal policy $\mathbf{x}^*_{it}$ given her beliefs, $\bar{\boldsymbol{\beta}}_{it}$.

2. Given $\mathbf{x}^*_{it}$, $y_t = \boldsymbol{\beta}_i'\mathbf{x}^*_{it} + \varepsilon_t$ is realized (and utility U_t gained). Both types update their beliefs using OLS. Mean beliefs evolve to $\bar{\boldsymbol{\beta}}_{j(t+1)}$, for all $j \in \{S, C\}$.

3. Type S (C) wins the election at period $t+1$ iff $\bar{\boldsymbol{\beta}}'_{s(t+1)}\bar{\boldsymbol{\beta}}_{s(t+1)} > (<)\ \bar{\boldsymbol{\beta}}'_{c(t+1)}\bar{\boldsymbol{\beta}}_{c(t+1)}$. In the case of equal intensities, some tie breaking rule determines the winner.[10]

Note that while the model of S is misspecified, this type, by using OLS estimation, still uses Bayes rule "rationally" to learn and update her beliefs. Crucially, as each type learns from the observed actions which depend on the endogenously evolving beliefs, the data observed at time t, both past policies and outcomes, are not iid overtime.

While in the above model we assume myopic choice of policies each period, the exogenous noise $\mathbf{n}$ "mimics" low-cost experimentation that allows both types to learn better within the prism of their subjective model. We also discuss extensions to this assumption in Section IV.

III Perpetual political cycles and extremism

In this section we present Theorem 1, our main result, characterizing the unique steady state the dynamic model converges to. The steady state involves political cycles and extreme policies espoused and implemented by type S.

To formalize the notion of political cycles, let θ_{jt} denote the share of time that $j \in \{S, C\}$ had been in power up to period t. We then have (for the proof see Appendix I):

Theorem 1: *For sufficiently small* σ_n^2*, the polity converges in probability to a unique equilibrium in which: (i)* ***Political cycles:*** $\theta_{st} \xrightarrow{p} \theta_s$, $0 < \theta_s < 1$, *(ii)* $\bar{\boldsymbol{\beta}}_{ct} \xrightarrow{p} \bar{\boldsymbol{\beta}}_c = \boldsymbol{\beta}$, *(iii)* ***Colinear and extreme beliefs for*** $S : \bar{\boldsymbol{\beta}}_{st} \xrightarrow{p} \bar{\boldsymbol{\beta}}_s = (\tau^*)\boldsymbol{\beta}_s$ *where* $\tau^* = \sqrt{\frac{\boldsymbol{\beta}'\boldsymbol{\beta}}{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}} > 1$.

The asymptotic equilibrium involves perpetual political cycles in which power changes hands between the two types, apart from equilibrium paths of measure zero. As we now illustrate, the dynamics of belief updating imply that the type in opposition becomes more and more intense about taking office vis a vis the type that is currently in power. In addition, the simple type's beliefs and prescribed actions converge to be colinear and more extreme than of those that the complex type espouses on the set of policies both deem relevant.

Below we provide an intuition for Theorem 1 in two steps. We first assume that beliefs and the share of time that S is in power, θ_{st}, converge and characterize the steady state. We then delve into the more technical discussion of what is involved in proving convergence of beliefs and θ_{st}.

[10] The exact tie breaking rule is inconsequential.

III.1 Characterizing the steady state

In this section we assume that beliefs, $\bar{\boldsymbol{\beta}}_{st}$, $\bar{\boldsymbol{\beta}}_{ct}$, and the share of time that S is in power, θ_{st}, converge. First, given that type C has the correct model and given the policy implementation noise, type C should converge to know the true parameters of the model, i.e., $\bar{\boldsymbol{\beta}}_{ct} \xrightarrow{p} \boldsymbol{\beta}$. This is shown formally in Appendix I and below we maintain this as an assumption.

We now focus on the asymptotic beliefs of S, $\bar{\boldsymbol{\beta}}_{st}$, as well as on the limit values of θ_{st}, θ_{ct} (where $\theta_{st} + \theta_{ct} = 1$). Let $\mathbf{x}^*_{sj}$ denote the limit of vector of chosen policies when type j is in power that S finds relevant. We will denote the full k x 1 vector of limit policies that C implements by $\mathbf{x}^*_c$.

Given convergence, the OLS coefficients converge to satisfy the following equation:[11]

$$(III.1) \quad \theta_s \mathbf{x}^*_{ss}(\mathbf{x}^{*\prime}_{ss}\bar{\boldsymbol{\beta}}_s - \mathbf{x}^{*\prime}_{ss}\boldsymbol{\beta}_s) + \theta_c \mathbf{x}^*_{sc}(\mathbf{x}^{*\prime}_{sc}\bar{\boldsymbol{\beta}}_s - \mathbf{x}^{*\prime}_{c}\boldsymbol{\beta}) = \sigma^2_n(\boldsymbol{\beta}_s - \bar{\boldsymbol{\beta}}_s),$$

where $(\mathbf{x}^{*\prime}_{ss}\bar{\boldsymbol{\beta}}_s - \mathbf{x}^{*\prime}_{ss}\boldsymbol{\beta}_s)$ and $(\mathbf{x}^{*\prime}_{sc}\bar{\boldsymbol{\beta}}_s - \mathbf{x}^{*\prime}_{c}\boldsymbol{\beta})$ are the average mistakes of type S, when S is in power and when C is in power respectively.

We first provide intuition for the result of perpetual cycles, which implies that $0 < \theta_s = 1 - \theta_c < 1$. To see this, suppose first that S is in power indefinitely, i.e. that $\theta_s = 1$. In this case, $(III.1)$ implies that S learns the true parameters $\boldsymbol{\beta}_s$; intuitively, in this case S has the correctly specified model and a small amount of noise guarantees true learning. However, given that it ignores some relevant policies, we have

$$(III.2) \quad \sqrt{\bar{\boldsymbol{\beta}}'_s\bar{\boldsymbol{\beta}}_s} = \sqrt{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s} < \sqrt{\boldsymbol{\beta}'\boldsymbol{\beta}},$$

which is a contradiction to the supposition that S is in power indefinitely as given that $\bar{\boldsymbol{\beta}}_c = \boldsymbol{\beta}$, $(III.2)$ implies that C have a higher intensity.

Suppose now that the polity converges so that C are in power indefinitely, i.e., $\theta_c = 1$. Now we make use of the fact that the level of noise is not too large. When σ^2_n is small, it is easy to see that in the solution to $(III.1)$, $(\mathbf{x}^{*\prime}_{sc}\bar{\boldsymbol{\beta}}_s - \mathbf{x}^{*\prime}_{c}\boldsymbol{\beta})$ will have to be small, converging to zero as σ^2_n shrinks to zero. Intuitively, with small noise and the same policies being chosen over time, type S will learn to predict expected outcomes y, i.e., for small σ^2_n,

$$(III.3) \quad \mathbf{x}^{*\prime}_{sc}\bar{\boldsymbol{\beta}}_s \simeq \mathbf{x}^{*\prime}_{c}\boldsymbol{\beta}.$$

Note however that the optimal action of S is $\mathbf{x}^*_s$ rather than $\mathbf{x}^*_{sc}$. Therefore, type S believes she could generate a strictly higher expected outcome if she was in power as S can take all resources from $C's$ policies that she does not believe to be relevant and add them to the

[11] This is the first order condition derived when minimizing expected squared mistakes; recall that σ^2_n is the variance of noise in policy implementation.

policies she deems relevant. As a result we have:

$$(III.4)\ \mathbf{x}_s^{*\prime}\bar{\boldsymbol{\beta}}_s > \mathbf{x}_{sc}^{*\prime}\bar{\boldsymbol{\beta}}_s$$

Combining $(III.3)$ and $(III.4)$, we have that,

$$(III.5)\quad \mathbf{x}_s^{*\prime}\bar{\boldsymbol{\beta}}_s > \mathbf{x}_c^{*\prime}\boldsymbol{\beta}.$$

Noting that $\mathbf{x}_s^* = \frac{\bar{\boldsymbol{\beta}}_s}{\sqrt{\bar{\boldsymbol{\beta}}_s'\bar{\boldsymbol{\beta}}_s}}\sqrt{R}$, and hence $\bar{\boldsymbol{\beta}}_s'\mathbf{x}_s^* = \sqrt{\bar{\boldsymbol{\beta}}_s'\bar{\boldsymbol{\beta}}_s}\sqrt{R}$, and similarly that $\mathbf{x}_c^{*\prime}\boldsymbol{\beta} = \sqrt{\boldsymbol{\beta}'\boldsymbol{\beta}}\sqrt{R}$, $(III.5)$ implies,

$$\begin{aligned}(III.6)\ \sqrt{\bar{\boldsymbol{\beta}}_s'\bar{\boldsymbol{\beta}}_s}\sqrt{R} &= \mathbf{x}_s^{*\prime}\bar{\boldsymbol{\beta}}_s > \mathbf{x}_c^{*\prime}\boldsymbol{\beta} = \sqrt{\boldsymbol{\beta}'\boldsymbol{\beta}}\sqrt{R} \Rightarrow \\ \sqrt{\bar{\boldsymbol{\beta}}_s'\bar{\boldsymbol{\beta}}_s} &> \sqrt{\boldsymbol{\beta}'\boldsymbol{\beta}},\end{aligned}$$

which contradicts the supposition that C is in power, as the intensity of preferences of S is higher.[12]

Thus, we must have $0 < \theta_s < 1$, and the equilibrium must satisfy equal intensity of preferences, or:

$$(III.7)\quad \sqrt{\bar{\boldsymbol{\beta}}_s'\bar{\boldsymbol{\beta}}_s} = \sqrt{\boldsymbol{\beta}'\boldsymbol{\beta}}.$$

Intuitively, when one group is in power indefinitely, the two types can in the long run get close to understanding the mean effects of policies. But this implies, given their different subjective models, that the two types have different beliefs, which leads to greater intensity of preferences for the type in opposition. Specifically, if C is in power, S's beliefs will suffer from an omitted variable bias; as a result, S believes it can increase its utility by gaining power and inflating policies. When on the other hand S is in power, it is C that gains higher intensity of preferences as it has additional parameters it believes to be effective, and thus knows it can improve the outcome as well.

We next show that type S's beliefs must be colinear with those of type C on the relevant shared policies. Moreover we show that on these shared policies, S espouses more extreme policies.

Note that the linear relation between $S's$ optimal actions and those of C have implications to minimizing expected mistakes $(\mathbf{x}_{ss}^{*\prime}\bar{\boldsymbol{\beta}}_s - \mathbf{x}_{ss}^{*\prime}\boldsymbol{\beta}_s)$ and $(\mathbf{x}_{sc}^{*\prime}\bar{\boldsymbol{\beta}}_s - \mathbf{x}_c^{*\prime}\boldsymbol{\beta})$. Suppose first that the steady state actions of S are *not* colinear with those of C. For small σ_n^2,this implies, from $(III.1)$, that the solution will involve that the expected mistakes of S in each regime, $(\mathbf{x}_{ss}^{*\prime}\bar{\boldsymbol{\beta}}_s - \mathbf{x}_{ss}^{*\prime}\boldsymbol{\beta}_s)$ and $(\mathbf{x}_{sc}^{*\prime}\bar{\boldsymbol{\beta}}_s - \mathbf{x}_c^{*\prime}\boldsymbol{\beta})$, are small and are close to zero (non colinearity of actions

[12] If the noise level is too large, then S will learn the truth about the parameters it considers, and C may remain in power indefinitely.

makes this possible).[13] So for small σ_n^2 we have,

$$(III.8a)\ \ \mathbf{x}_{ss}^{*\prime}\bar{\boldsymbol{\beta}}_s - \mathbf{x}_{ss}^{*\prime}\boldsymbol{\beta}_s \simeq \mathbf{0}$$
$$(III.8b)\ \ \mathbf{x}_{sc}^{*\prime}\bar{\boldsymbol{\beta}}_s - \mathbf{x}_{c}^{*\prime}\boldsymbol{\beta} \simeq \mathbf{0}$$

However, similar to our arguments above following from $(III.3)$, $(III.8b)$ contradicts equal intensity and cannot arise in equilibrium. We therefore conclude that beliefs and policies must be colinear. In other words, S cannot learn too much in equilibrium: Equilibrium policies have to be colinear to limit the learning of S and specifically her ability to predict expected output at each regime.

To see why S will hold more extreme beliefs than type C remember that in the long run the two types have equal intensity, i.e., $\bar{\boldsymbol{\beta}}_s'\bar{\boldsymbol{\beta}}_s = \boldsymbol{\beta}'\boldsymbol{\beta}$. Combining the colinearity result, so that $\bar{\boldsymbol{\beta}}_s = \tau\boldsymbol{\beta}_s$ for some τ, and the equal intensity condition, we pin down the equilibrium degree of colinearity τ^* :

$$(III.9)\ (\tau^*)^2(\boldsymbol{\beta}_s'\boldsymbol{\beta}_s) = \boldsymbol{\beta}'\boldsymbol{\beta} \Rightarrow \tau^* = \sqrt{\frac{\boldsymbol{\beta}'\boldsymbol{\beta}}{\boldsymbol{\beta}_s'\boldsymbol{\beta}_s}} > 1 \Rightarrow$$
$$\bar{\boldsymbol{\beta}}_s = \sqrt{\frac{\boldsymbol{\beta}'\boldsymbol{\beta}}{\boldsymbol{\beta}_s'\boldsymbol{\beta}_s}}\boldsymbol{\beta}$$

The collinearity result implies that S is more bold in its policy prescriptions, and that both groups agree on the relative effectiveness of the policies that they both consider relevant. Therefore, in our model simplicity implies extremism.

From $(III.9)$ we see that the more important are the parameters that S ignores, relative to those she considers, the more extreme are her beliefs, as well as policies. As we show, this will imply a lower equilibrium value for θ_s. Intuitively, to generate more extreme beliefs in equilibrium, S needs to suffer from a higher omitted variable bias, which arises when C is in power more often. Thus, political cycles must result in just enough omitted variable bias to equate intensity. Specifically, to solve for θ_s, we plug the expression for $\bar{\boldsymbol{\beta}}_s$ from $(III.9)$ in $(III.1)$ and get:

$$\theta_s = \frac{1 - \tau^* \frac{\sigma_n^2}{R}}{1 + \tau^*},$$

where it is easy to see that θ_s is lower when τ^* is higher. The following observation summarizes the above discussion:

Observation 1: *The more important are the policy variables that S ignores, the more extreme are S's belief, and the less time it spends in power.*

[13] When $\sigma_n^2 = 0$, S would be able to conjecture correctly the average output *at each regime*. In other words, non colinearity implies that S can solve $(III.1)$ by solving both equations below as they are independent of one another.

Note that in equilibrium, when S is in power, S is on average disappointed in its policies. On the other hand whenever C is in power, type S is positively surprised. The following observation characterizes the expected mistakes of S in the different regimes:

Observation 2: *In the limit:*

$$(III.11) \quad \mathbf{x}_s^{*\prime}\bar{\boldsymbol{\beta}}_s > \mathbf{x}_s^{*\prime}\boldsymbol{\beta}$$

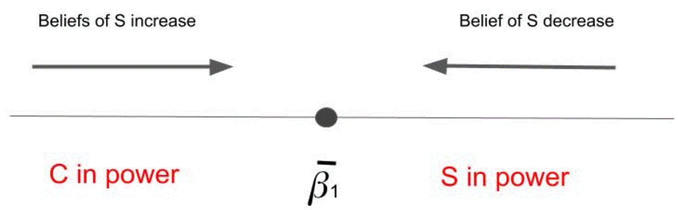
$$(III.12) \; \mathbf{x}_{sc}^{*\prime}\bar{\boldsymbol{\beta}}_s < \mathbf{x}_c^{*\prime}\boldsymbol{\beta}.$$

These two inequalities imply the long-term dynamics of S's beliefs and policies, where they moderate when in power but become more intense when in opposition. To illustrate this graphically, consider a simple one-dimensional example where the true model is $y = \beta_1 x_1 + \beta_2 x_2 + \varepsilon$, but S believes that $\beta_2 = 0$ so that only x_1 is relevant.[14] The equal intensity condition pins down the belief of S as follows:

$$(III.13) \quad \bar{\beta}_1 = \sqrt{(\beta_1)^2 + (\beta_2)^2}$$

The figure below describes the asymptotic belief of S, close to the equilibrium above, when C had already converged to the truth. Close to the equal intensity belief, whenever the intensity of preferences of S is larger than that of C, it gains power and implements its ideal policy. But then, on average, S becomes disappointed in the outcomes it generates and moderates its belief. Simple voters are systematically disappointed by the outcomes of the extreme policies implemented when their populist politicians are in power. This leads to a gradual diminution of beliefs and consequent moderation of policy, until those with more complex views once again take power. Whenever $S's$ intensity falls below that of C, and C gains power, S starts to inflate the effectiveness of x_1. The surprising success of policy under the complex gradually convinces simple voters of the value of implementing more extreme and focused policies, increasing their probability of voting in favour of populist politicians who advocate narrow and extreme solutions to complex problems. The equal intensity belief is then a basin of attraction for this dynamics.

[14]In this one-dimensional case colinearity is trivially satisfied.



In Section IV we report additional comparative statics results in terms of the effect of good and bad ε shocks on the speed of power shifts.

III.2 Convergence

In general, establishing convergence with misspecified models is problematic even with exogenous iid data (see Berk 1966). Having endogenous data, as we have in our model, introduces more challenges as observations are non iid. As we mentioned in the introduction, substantial progress has been made in the literature analyzing the convergence properties of misspecified models with non iid data.[15] But with respect to this literature, our model is further complicated by having multiple players, continuous actions, and multidimensional state space.

Specifically, multiple dimensions of policy allows for the possibility that types entertain multiple equilibrium beliefs in the long term. This multiplicity introduces additional challenges for establishing convergence as it is hard to prove that types do not perpetually "travel" along this continuum of beliefs. As we show below, the policy noise, $\mathbf{n}$, allows us to establish convergence in this model.

In the appendix we prove convergence with the following steps. First, we establish a law of large numbers for our framework that relies on the fact that at period t, the regressors $\mathbf{x}_t$ and the shock ε_t are independent of each other. While the regressors depend on past realizations of the shock, they are not correlated with the current one. This law of large numbers allows us to show that the beliefs of C converge, with the help of the noise $\mathbf{n}$, to the true parameters and so $\bar{\boldsymbol{\beta}}_c = \boldsymbol{\beta}$. Given these two steps we can derive a *deterministic* law

[15]See for example Esponda, Pouzo and Yamamaoto (2019) and Frick, Iijima and Ishii (2020).

of motion for the asymptotic beliefs of S.

$$(III.14)\ \bar{\boldsymbol{\beta}}_s \overset{p}{\to} \boldsymbol{\beta}_s + c\mathbf{M}^{-1}\boldsymbol{\beta}_s, \text{ where}$$
$$\mathbf{M} = \frac{\mathbf{X}'_{ss}\mathbf{X}_{ss}}{t_c R} + \frac{t_c + t_s}{t_c}\frac{\sigma_n^2}{R}\mathbf{I}_{k_s} + \frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'_s}{\boldsymbol{\beta}'\boldsymbol{\beta}},\ c = 1 - \frac{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}{\boldsymbol{\beta}'\boldsymbol{\beta}}$$

where $\mathbf{X}_{ss}$ denotes the matrix of regressors that S find relevant and have been implemented when S has been in power, and t_i denotes the number of periods type i has been in power up to period t (so that $t_i/t = \theta_{it}$).

The policy noise allows S to learn the true relative merits of each policy. Moreover, as long as S's policies are not colinear with those of C, its policies also provide enough variation in the data so that S can learn the true relative merits of the policies they focus on. This implies that beliefs and policies converge to be colinear. The omitted variable bias captured above by $c\mathbf{M}^{-1}\boldsymbol{\beta}_s$ shifts up and down until asymptotic power sharing results in just enough bias to reach equal intensity, and the dynamics of this are similar to those described in the one-dimensional case.[16]

IV Additional results and discussion

In this Section we present some additional results and discuss alternative modelling assumptions.

IV.1 Relation to Berk-Nash equilibrium

In this section we examine the relation between our results above and a static notion of equilibrium in the spirit of Berk-Nash equilibrium (Esponda and Pouzo 2016). We focus on a more general Bayesian framework and assume that there is no policy noise. In particular, we maintain all the assumptions of the model above but assume more generally that: (i) the prior on $\boldsymbol{\beta} \in \mathbb{R}^k$ is not necessarily normal, (ii) updating follows Bayesian updating and the distribution of the shocks is governed by (commonly known) $f(\varepsilon)$, which is a continuous and differentiable density on $\mathbb{R}$, and satisfies the some boundedness conditions as in Berk (1966), so that the minimum Kullback-Leibler distance below exists,[17] (iii) $\sigma_n^2 = 0$.

A Berk-Nash equilibrium is a static solution concept for a dynamic game of players with misspecified models where actions are optimal given beliefs and beliefs rationalize the observed output which arises given the actions played. Berk (1966) shows for the case of iid data that beliefs stemming from a misspecified model will concentrate on those that minimize

[16] The Appendix illustrates the phase diagrams derived from $(III.14)$.

[17] These are conditions (iii) and (iv) in Berk (1996), referred to in Lemma 2 in that paper that proves the existence of the minimum of Kullback-Leibler distance.

the Kullback–Leibler (KL) distance to the true beliefs.[18] Using this notion of minimizing the KL distance, Esponda and Pouzo (2016) define a Berk-Nash equilibrium.

Here we adopt the Berk-Nash solution concept to our model.[19] An important part of this definition is the parameter $\theta_s \in [0,1]$ which denotes the probability that type S is in power. In the dynamic interpretation of this equilibrium, analyzed above, θ_s captures the fraction of time that S was in power.

Definition 1: A Berk-Nash equilibrium consists of beliefs for $i \in \{S, C\}$ with mean $\bar{\boldsymbol{\beta}}_i$, a policy choice $\mathbf{x}_i$, and a probability that type S is in power, $\theta_s \in [0,1]$, such that:

$(IV.1)$ **Optimal actions:** $\mathbf{x}_i$ is the optimal action given mean beliefs $\bar{\boldsymbol{\beta}}_i$ and so $\mathbf{x}_i = \mathbf{x}_i^*$.

$(IV.2)$ **Power sharing according to intensity**: $\theta_s = 1$ (0) if $\bar{\boldsymbol{\beta}}_s'\bar{\boldsymbol{\beta}}_s > (<)$ $\bar{\boldsymbol{\beta}}_c'\bar{\boldsymbol{\beta}}_c$; if $\bar{\boldsymbol{\beta}}_s'\bar{\boldsymbol{\beta}}_s = \bar{\boldsymbol{\beta}}_c'\bar{\boldsymbol{\beta}}_c$, $\theta_s \in [0,1]$.

$(IV.3)$ **Beliefs minimize Kullback–Leibler distance**: Given actions $\mathbf{x}_c$,$\mathbf{x}_s$ and θ_s, each vector in the support of i's beliefs solves, according to their subjective model:

$$\min_{\hat{\boldsymbol{\beta}}_i} E_\varepsilon[\theta_s \ln \frac{f(\varepsilon)}{f(\boldsymbol{\beta}_s'\mathbf{x}_s - \hat{\boldsymbol{\beta}}_i'\mathbf{x}_{is} + \varepsilon)} + (1-\theta_s) \ln \frac{f(\varepsilon)}{f(\boldsymbol{\beta}'\mathbf{x}_c - \hat{\boldsymbol{\beta}}_i'\mathbf{x}_{ic} + \varepsilon)}]$$

We first show that an equilibrium analogous to the one identified in Theorem 1 is a Berk-Nash equilibrium of the more general model (proofs for the results in this Section are in Appendix II):

Proposition 1: *There exists a Berk-Nash equilibrium with* $\bar{\boldsymbol{\beta}}_c = \boldsymbol{\beta}$, $\bar{\boldsymbol{\beta}}_s = \tau^*\boldsymbol{\beta}_s$ *and* $0 < \theta_s < 1$. *In this equilibrium, when* f *is normal,* $\theta_s = \frac{1}{1+\tau^*} = \lim_{\sigma_n^2 \to 0} \frac{1-\tau^* \frac{\sigma_n^2}{R}}{1+\tau^*}$.

The proof of Proposition 1 follows similar arguments to those in section III using the notions of expected mistakes under the subjective models. To see this suppose for example that C is in power with probability one, i.e., $\theta_s = 0$. By $(IV.3)$ each vector $\hat{\boldsymbol{\beta}}_s$ in the support of $S's$ beliefs must minimize

$$(IV.4) \quad E_\varepsilon \ln(\frac{f(\varepsilon)}{f(\boldsymbol{\beta}'\mathbf{x}_c^* - \hat{\boldsymbol{\beta}}_s'\mathbf{x}_{sc}^* + \varepsilon)})$$

By Gibb's inequality, the Kullback–Leibler divergence is larger or equal to zero, holding with equality if and only if both densities, the true density and the subjective density, coincide almost everywhere. This implies that $(IV.4)$ is minimized at $\boldsymbol{\beta}'\mathbf{x}_c^* = \hat{\boldsymbol{\beta}}_s'\mathbf{x}_{sc}^*$ for each $\hat{\boldsymbol{\beta}}_s$ in the support. By linearity, this implies that the mean beliefs of S also satisfy:

$$(IV.5) \quad \boldsymbol{\beta}'\mathbf{x}_c^* = \bar{\boldsymbol{\beta}}_s'\mathbf{x}_{sc}^*$$

[18]Intuitively, minimising the Kullback–Leibler distance is similar to maximising the likelihood of previous observations.

[19]Our model is not formally a game, which is why we cannot use the definition of Esponda and Pouzo (2016) directly.

and the argument follows directly from the argument we have made in Section III.

But the definition above allows for multiplicity of Berk-Nash equilibria. Consider for example an equilibrium configuration in which both types hold exactly the same average beliefs. Suppose further that these are the true parameter values for all the policies that S's subjective model deems relevant and zero average beliefs on all other policy parameters. Thus, C "abandons" some relevant policy variables that were included in her prior belief. It is easy to see that this configuration, together with any θ_s in $[0, 1]$, constitutes a Berk-Nash equilibrium. In particular, C is satisfied with believing that some policies are irrelevant, because in equilibrium these policies are never played. In section III, the policy noise ensured that C did not "abandon" any relevant policies and hence the equilibrium was unique.

Even though there are multiple equilibria, we show that policy inefficiency is an inherent feature of any Berk-Nash equilibrium:

Proposition 2 (Inefficiency of political competition): *Any Berk-Nash equilibrium will involve inefficient policy implementation with a strictly positive probability. In particular, any equilibrium will be characterized either by $\theta_s > 0$ or by C having zero expected beliefs on some of its relevant policies.*

IV.2 More general subjective models

Above we considered an environment in which the beliefs of "complex" types are correctly specified, in that they include all relevant policies, whereas "simple" types erroneously exclude a subset of these. In Appendix I we consider an extension in which both types can also consider policies which are irrelevant. Specifically, we maintain that the simple type considers a subset of the relevant policies that the complex type considers, but assume that the prior beliefs of both types may also include some irrelevant policies that have zero effects. We impose no *a priori* restriction on the relative number of policies each type believes may be relevant. That is, it may be that the "complex" type overall considers a smaller number of policies to be relevant.

We show that the endogenous asymptotic equilibrium looks much like the one in our basic model and so the "simple" and "complex" tags arise endogenously. That is, we show that the beliefs of both types regarding policies that are actually irrelevant converge on 0 (this arises through our use of small policy implementation noise). Consequently, the non-zero beliefs of those with the misspecified model become "simple" relative to the "complex" views of those with the correctly specified model. While the beliefs of the complex converge on true parameter values, the beliefs of the simple converge on a multiple of the true parameter values, as in Theorem 1.

IV.3 Local dynamics: Random outcomes and the political cycle

A peculiar characteristic of political life seems to be that random outcomes benefit or harm incumbent parties. In the online appendix we show that this feature arises in our model through the fully rational Bayesian updating of beliefs. Random shocks change estimates of the effective- ness of policy, but these effects are stronger for the incumbent party which is implementing its desired policy combination.

We focus on outcomes close to the steady state and, to simplify the analysis, with negligible amounts of policy noise. We show that close to the steady state a random negative ε shock to y lowers the relative intensity of the incumbent group, hastening regime change, while random positive ε shocks to y strengthen the relative intensity of the incumbent group, lengthening their stay in power in the current political cycle.

Specifically, when the simple group is in power, a negative shock reduces their intensity, as their belief in the effectiveness of the policies they deem relevant falls. Complex beliefs in these same policies also fall, but the complex belief in the efficacy of policies the simple deem irrelevant, and hence do not implement, rises, as the poor outcome under simple rule convinces the complex that these neglected policies are more effective than previously thought. These two effects offset each other, and complex intensity remains constant. In sum, a negative shock lowers the relative political intensity of the simple, hastening the transfer of power, with positive shocks having the opposite effect.

When the complex are in power, a negative shock reduces the belief in the effectiveness of policies of both types, but the effects on intensity are greater for the complex, for whom intensity depends upon a wider range of policies, all of which are seen to be failing. Consequently, negative shocks accelerate regime change, ushering in further negative outcomes as the simple implement misguidedly narrow and intense policies, while positive shocks lengthen the time the complex hold onto power and the polity continues to benefit from a full range of moderate policy actions.

IV.4 Endogenous resource constraints

In our model we have assumed a fixed resource constraint R. We can extend the model to allow the different types to endogenously choose their desired level of resources. In particular, we can assume that the utility citizens derive from all common outcomes is given by:

$$U_t = y_t + V(R_t),$$

where as before $R_t = \mathbf{x}_t'\mathbf{x}_t$ represents the resources used in implementing policy $\mathbf{x}_t$ for y_t, while V represents the utility derived from policy outcomes over which there is no disagreement regarding causal mechanisms. V is a reduced form, representing the utility that can be achieved in other policy areas given the allocation of resources to y_t, and the assumptions $V' < 0$ and $V'' < 0$ are natural. To derive analytical results, we work with a second-order approximation of V as a quadratic function of R_t. We can then show that intensity of preferences is also an increasing function of the magnitude of beliefs. Assuming that R_t is bounded from above, we can then extend all our convergence results accordingly.

IV.5 Discussion of other assumptions

Our framework is one of model misspecification and as such needs to take a stand on the nature of the misspecification. We have chosen to focus on linear misspecified models for several reasons. First, as our focus is on simplistic versus more intricate world views, linear models that differ in terms of the set of free parameters allow us to define simplicity in a straightforward way. Non-linear models will surely introduce more difficulty in formalizing a notion of simplicity. Moreover, the linear structure of the misspecified model, together with the quadratic resource constraints, allow us to easily calculate and analyze the notion of intensity of preferences which is the main driver of the political force in the model.

We also assume a simple utility function that is linear in y, which implies that utility is a function of mean beliefs only. For more general utilities the whole distribution of beliefs would matter. Montiel Olea et al (2017) show that in a model with exogenous data, complex models (which abide with the truth) would induce lower variance of their beliefs when data is sufficiently large. This would imply an advantage to the complex group. Thus, our results hold as long as individuals are not too risk averse.

In terms of our political model, we assume that the winning politician implements her myopic ideal policy; that is, she does not experiment in order to learn or to manipulate future actions and political outcomes. Intuitively it is more difficult to woe voters with sophisticated long term policies as compared to just sticking to the myopic ideal policy. To a degree, the addition of noise captures some form of experimentation. More sophisticated forward strategic behavior, with the purpose of manipulating the actions and outcomes of future periods, is beyond the scope of our analysis. Such a possibility may potentially affect the political cycle result we report, but we conjecture that the views advocated by those with a simple model will still affect political outcomes.

We use a simple political model in which intensity of preferences is the key to electoral success. We have two groups, and we adopt a citizen candidate model so that politicians offer voters exactly their ideal polices. One may imagine other models of political competition, e.g., probabilistic voting with office motivated politicians, which essentially implies that politicians choose policies to maximize average welfare. While this would yield different policies as well as learning patterns, a key feature of our analysis will remain: In equilibria, policies will cater to group S to some degree. That is, the omitted variable bias in $S's$ beliefs would mean that they would prefer stronger policies on the policies they deem effective. Any policy that maximizes welfare will then exhibit such a bias.

V Conclusion

Our analysis has shown how simplistic beliefs can persist in political competition against a more accurate and complex view of the world, delivering sub-par outcomes on each outing in power and yet returning to dominate the political landscape over and over again. In the framework presented above simplistic beliefs arise as a consequence of a primitive assumption of misspecification, but we recognize that there are deeper questions to explore. A recent examination of European Social Survey data by Guiso et al (2017) finds that the responsiveness of the electorate to populist ideas and the supply of populist politicians increases in periods of economic insecurity. Social and economic transformation, and the insecurity and inequality it can engender, may create environments in which opportunistic politicians are able to plant erroneously simplistic world views into the electorate. Linking belief formation, at its most fundamental level, to ongoing economic and political events allows a richer characterization of political cycles, and is something we intend to explore in future work.

Appendix I: Proof of Convergence in a Generalized Model (a generalisation of Theorem 1)

In this appendix we prove convergence to the probability limits for beliefs and the share of time each type is in power given in the paper in a generalized framework. Specifically, while in the paper all k potential policies were relevant (i.e. had non-zero effects), in this appendix we allow that some may be irrelevant and have zero effects. While the beliefs of "complex" types are correctly specified, in that they include all relevant policies, "simple" types erroneously exclude a subset of these. The prior beliefs of both types may include some irrelevant policies that have zero effects, and we impose no a priori restriction on the relative number of policies, k_s and k_c, each type believes may be relevant, other than that their union covers the set of k policies that are systematically implemented. The monikers "complex" and "simple" derive from the fact that the endogenous asymptotic equilibrium looks much like that assumed in the paper, where the non-zero beliefs of the complex are broader than those of the simple.

We begin by reviewing notation. $\mathbf{H} = \mathbf{X} + \mathbf{N}$ denotes the t x k history of policy and noise, $\mathbf{H}_i$ and $\mathbf{H}_{\sim i}$ the k_i and $k_{\sim i}$ columns of that history deemed relevant and irrelevant by type i, and $\boldsymbol{\beta}$, $\boldsymbol{\beta}_i$ and $\boldsymbol{\beta}_{\sim i}$ the true values of the parameters and the subsets of these associated with the policies type i believes are and are not relevant. $\mathbf{H}_{ij}$ and $\mathbf{H}_{\sim ij}$ are the rows of $\mathbf{H}_i$ and $\mathbf{H}_{\sim i}$ associated with the t_j periods when type j is in power, with $t_i + t_j = t$. We use the notation $\mathbf{H}_{\bullet j}$ to denote the t_j x k history of all policies during the periods type j is in power. $\mathbf{I}_k$ and $\mathbf{0}_{nxm}$ denote the identity matrix and matrix of zeros of the indicated dimensions.

(A) Preliminaries: Standard Matrix Algebra Results & Some Lemmas

A symmetric positive definite matrix $\mathbf{V}$ allows the spectral decomposition $\mathbf{E\Lambda E'}$, where $\boldsymbol{\Lambda}$ is the diagonal matrix of strictly positive eigenvalues and $\mathbf{E}$ is a matrix whose columns are the corresponding mutually orthogonal eigenvectors, with $\mathbf{EE'} = \mathbf{E'E} = \mathbf{I}$. $\mathbf{V}^{-1} = \mathbf{E\Lambda}^{-1}\mathbf{E'}$, i.e. the inverse of $\mathbf{V}$ has the same eigenvectors as $\mathbf{V}$ and eigenvalues equal to the inverse of those of $\mathbf{V}$. We can also define $\mathbf{V}^{-½} = \mathbf{E\Lambda}^{-½}\mathbf{E'}$ as $\mathbf{V}^{-½}\mathbf{V}^{-½} = \mathbf{E\Lambda}^{-½}\mathbf{E'E\Lambda}^{-½}\mathbf{E'} = \mathbf{E\Lambda}^{-1}\mathbf{E'}$. In a similar spirit, $\mathbf{V}^{-2} = \mathbf{V}^{-1}\mathbf{V}^{-1}$ has eigenvalues equal to the square of those of $\mathbf{V}^{-1}$ and the same eigenvectors. For a rank one update of $\mathbf{V}$ using the vector $\mathbf{x}$, the Sherman-Morrison formula tells us that $(\mathbf{V}+\mathbf{xx'})^{-1}=\mathbf{V}^{-1}-\mathbf{V}^{-1}\mathbf{xx'V}^{-1}/(1+\mathbf{x'V}^{-1}\mathbf{x})$, while the eigenvalues of the matrix ($\mathbf{V}$ + $c\mathbf{I}$), with c a constant, are given by $\mathbf{\Lambda}+c\mathbf{I}$, and the eigenvectors are the same as those of $\mathbf{V}$. The eigenvalues of $\mathbf{V}$ are all weakly increasing following a rank-one update (Golub 1973), so if $\mathbf{V}$ is initially positive definite (with strictly positive eigenvalues) it remains so following a sequence of rank-one updates. The maximum across all possible vectors $\mathbf{x}$ of the Rayleigh quotient $\mathbf{x'Vx/x'x}$ is the maximum eigenvalue of $\mathbf{V}$, which we denote with $\lambda_{\max}(\mathbf{V})$, with $\lambda_{\min}(\mathbf{V})$ denoting the minimum eigenvalue.

The following two lemmas are used repeatedly in our proofs:

$$\text{(L1a)}\ \frac{\mathbf{X'\varepsilon}}{t}\xrightarrow{p}\mathbf{0}_{kx1};\ \text{(L1b)}\ \frac{\mathbf{X'N}}{t}\xrightarrow{p}\mathbf{0}_{kxk};\ \text{(L1c)}\ \frac{\mathbf{N'\varepsilon}}{t}\xrightarrow{p}\mathbf{0}_{kx1};\ \text{(L1d)}\ \frac{\mathbf{N'N}}{t}\xrightarrow{p}\sigma_n^2\mathbf{I}_k$$

$$\text{(L2)}\ \boldsymbol{\varepsilon}'\mathbf{H}_i'(\mathbf{H}_i'\mathbf{H}_i)^{-1}(\mathbf{H}_i'\mathbf{H}_i)^{-1}\mathbf{H}_i\boldsymbol{\varepsilon}\xrightarrow{p}0$$

where $\mathbf{I}_k$ and $\mathbf{0}_{ixj}$ denote the identity matrix and matrix of zeros of specified dimensions, $\xrightarrow{p}$ denotes "converges in probability to", and the t x k matrix $\mathbf{H}=\mathbf{X}+\mathbf{N}$ denotes the history of desired policy and noise, $\mathbf{H}_i$ the columns of that history viewed as relevant by type i, and $\boldsymbol{\varepsilon}$ the t x 1 history of the iid error in the realization of $\mathbf{y}$.

The i^{th} element of the vector (L1a) is

$$\text{(A1)}\ \sum_{n=1}^{t}\frac{x_{in}\varepsilon_n}{t}.$$

As each ε_n is independent of contemporaneous policy and past shocks and policy, applying the law of iterated expectations (i.e. taking the expectation at time 0 of the expectation at time 1 of the expectation at time 2 ...) one sees that

$$\text{(A2)}\ E\left[\sum_{n=1}^{t}x_{in}\varepsilon_n\right]=0,\quad E\left[\left(\sum_{n=1}^{t}x_{in}\varepsilon_n\right)^2\right]=\sum_{n=1}^{t}E(x_{in}^2)\sigma_\varepsilon^2,$$

As x_{in}^2 is bounded by the total resources devoted to policy (R) its expectation exists and is bounded:

$$\text{(A3)}\ E(x_{in}^2)=\int_0^\infty P(x_{in}^2\geq v)\,dv=\int_0^R P(x_{in}^2\geq v)\,dv\leq R.$$

Consequently, the average of the summation converges in mean square and hence in probability as well

$$\text{(A4)} \lim_{t\to\infty} E\left[\sum_{n=1}^{t} \frac{x_{in}\varepsilon_n}{t}\right] = 0 \ \& \lim_{t\to\infty} E\left[\left(\sum_{n=1}^{t} \frac{x_{in}\varepsilon_n}{t}\right)^2\right] \leq \lim_{t\to\infty} \frac{R\sigma_\varepsilon^2}{t} = 0 \Rightarrow \sum_{n=1}^{t} x_{in}\varepsilon_n / t \overset{p}{\to} 0,$$

which establishes (L1a). The i x j^{th} element of (L1b) is

$$\text{(A5)} \ \sum_{n=1}^{t} \frac{x_{in} n_{jn}}{t},$$

and since n_{jn} is an iid mean-zero finite-variance random variable, by the same logic as used in the proof of (L1a) this converges in probability to zero. Finally, the i^{th} element of (L1c) and i x j^{th} element of (L1d) are

$$\text{(A6)} \ \sum_{n=1}^{t} \frac{n_{in}\varepsilon_n}{t} \text{ and } \sum_{n=1}^{t} \frac{n_{in} n_{jn}}{t}.$$

The product of two iid and mutually independent random variables is an iid random variable in its own right, and hence by the strong law of large numbers these terms converge almost surely to their expectation, which proves (L1c) and (L1d). Cross-products based on the policies and noise each type believes are relevant, $\mathbf{X}_i$ and $\mathbf{N}_i$, are simply subsets of the results in (L1), and obviously follow the same probability limits.

Turning to (L2), we begin by noting that

$$\text{(A7)} \ \frac{\boldsymbol{\varepsilon}'\mathbf{H}_i}{t}\left[\frac{\mathbf{H}_i'\mathbf{H}_i}{t}\right]^{-1}\left[\frac{\mathbf{H}_i'\mathbf{H}_i}{t}\right]^{-1}\frac{\mathbf{H}_i'\boldsymbol{\varepsilon}}{t} \leq \frac{\boldsymbol{\varepsilon}'\mathbf{H}_i}{t}\frac{\mathbf{H}_i'\boldsymbol{\varepsilon}}{t}\lambda_{\max}\left(\left[\frac{\mathbf{H}_i'\mathbf{H}_i}{t}\right]^{-1}\left[\frac{\mathbf{H}_i'\mathbf{H}_i}{t}\right]^{-1}\right)$$

$$= \frac{\boldsymbol{\varepsilon}'\mathbf{H}_i}{t}\frac{\mathbf{H}_i'\boldsymbol{\varepsilon}}{t}\lambda_{\min}\left(\frac{\mathbf{H}_i'\mathbf{H}_i}{t}\right)^{-2} \leq \frac{\boldsymbol{\varepsilon}'\mathbf{H}_i}{t}\frac{\mathbf{H}_i'\boldsymbol{\varepsilon}}{t}\lambda_{\min}\left(\frac{\mathbf{H}_i'\mathbf{H}_i - \mathbf{X}_i'\mathbf{X}_i}{t}\right)^{-2}$$

where in the first inequality we use the properties of the Rayleigh quotient, in the following equality the relation between the eigenvalues of matrix products and inverses, and in the final inequality the fact that in the t rank one updates of matrix $\mathbf{H}_i'\mathbf{H}_i - \mathbf{X}_i'\mathbf{X}_i$ to $\mathbf{H}_i'\mathbf{H}_i$ the eigenvalues are always weakly increasing. Noting that $\mathbf{H}_i'\mathbf{H}_i - \mathbf{X}_i'\mathbf{X}_i$ = $\mathbf{X}_i'\mathbf{N}_i + \mathbf{N}_i'\mathbf{X}_i + \mathbf{N}_i'\mathbf{N}_i$ and applying the probability limits from (L1), we see that

$$\text{(A8)} \ \frac{\boldsymbol{\varepsilon}'\mathbf{H}_i}{t}\frac{\mathbf{H}_i'\boldsymbol{\varepsilon}}{t}\lambda_{\min}\left(\frac{\mathbf{H}_i'\mathbf{H}_i - \mathbf{X}_i'\mathbf{X}_i}{t}\right)^{-2} \overset{p}{\to} \mathbf{0}_{kx1}'\mathbf{0}_{kx1}\lambda_{\min}(\sigma_n^2\mathbf{I}_k)^{-2} = \frac{0}{(\sigma_n^2)^2} = 0.$$

Since $\boldsymbol{\varepsilon}'\mathbf{H}_i(\mathbf{H}_i'\mathbf{H}_i)^{-1}(\mathbf{H}_i'\mathbf{H}_i)^{-1}\mathbf{H}_i'\boldsymbol{\varepsilon}$ is a non-negative random variable bounded from above by a random variable whose probability limit is zero, it follows that (L2) is true.

Standard econometric proofs start off by assuming that the plim of $(\mathbf{H}_i'\mathbf{H}_i/t)^{-1}$ is a positive definite matrix, arguing that the plim of $\boldsymbol{\varepsilon}'\mathbf{H}_i/t$ is a vector of 0s, and then drawing conclusions about the plim of $(\mathbf{H}_i'\mathbf{H}_i)^{-1}\mathbf{H}_i'\boldsymbol{\varepsilon}$. In our case, since the regressors are endogenous, we cannot make a priori assumptions about whether the plim of $(\mathbf{H}_i'\mathbf{H}_i/t)^{-1}$ even exists. However, as (L2) shows, a quadratic form based upon $(\mathbf{H}_i'\mathbf{H}_i/t)^{-1}$ is easily shown to be bounded and to converge to zero provided there is minimal noise. In the proofs below we make use of such quadratic forms to prove that beliefs and other objects of interest converge.

(B) Convergence in the Generalized Model

The complex's model incorporates the effects of all policies whose effects are non-zero and their mean beliefs are given by

$$\text{(B1)}\quad \overline{\boldsymbol{\beta}}_c = (\mathbf{H}_c'\mathbf{H}_c)^{-1}\mathbf{H}_c'\mathbf{y} = (\mathbf{H}_c'\mathbf{H}_c)^{-1}\mathbf{H}_c'(\mathbf{H}_c\boldsymbol{\beta}_c + \mathbf{H}_{\sim c}\boldsymbol{\beta}_{\sim c} + \boldsymbol{\varepsilon}) = \boldsymbol{\beta}_c + (\mathbf{H}_c'\mathbf{H}_c)^{-1}\mathbf{H}_c'\boldsymbol{\varepsilon},$$

$$\text{so}\ (\overline{\boldsymbol{\beta}}_c - \boldsymbol{\beta}_c)'(\overline{\boldsymbol{\beta}}_c - \boldsymbol{\beta}_c) = \boldsymbol{\varepsilon}'\mathbf{H}_c(\mathbf{H}_c'\mathbf{H}_c)^{-1}(\mathbf{H}_c'\mathbf{H}_c)^{-1}\mathbf{H}_c'\boldsymbol{\varepsilon} \overset{p}{\to} 0,$$

where the first line uses the fact that all elements of $\boldsymbol{\beta}_{\sim c}$ are zero and the second line follows from Lemma 2 above. Consequently, we know that the beliefs of the complex converge on the true parameter values

$$\text{(B2)}\quad \overline{\boldsymbol{\beta}}_c \overset{p}{\to} \boldsymbol{\beta}_c,$$

and in the probability limit the complex implement policies

$$\text{(B3)}\quad \boldsymbol{\beta}\sqrt{R/\boldsymbol{\beta}'\boldsymbol{\beta}}$$

where R denotes the available resources and where we have used the fact that since the elements of $\boldsymbol{\beta}_{\sim c}$ are all zero we can express complex policies in the areas they believe are irrelevant in terms of these parameters as well. The remainder of this appendix is devoted to proving that simple beliefs $\overline{\boldsymbol{\beta}}_s$ converge on the steady state values $\tau^*\boldsymbol{\beta}_s$, where $\tau^* = \sqrt{\boldsymbol{\beta}'\boldsymbol{\beta}/\boldsymbol{\beta}_s'\boldsymbol{\beta}_s}$. We note that τ^* is strictly greater than 1, as we assume that simple beliefs are misspecified, so $\boldsymbol{\beta}_{\sim s} \neq \mathbf{0}_{k_{\sim s}x1}$.

The simple's mean beliefs are given by the coefficient estimates in the misspecified regression

$$\text{(B4)}\quad \overline{\boldsymbol{\beta}}_s = (\mathbf{H}_s'\mathbf{H}_s)^{-1}\mathbf{H}_s'\mathbf{y} = (\mathbf{H}_s'\mathbf{H}_s)^{-1}\mathbf{H}_s'\mathbf{H}\boldsymbol{\beta} + (\mathbf{H}_s'\mathbf{H}_s)^{-1}\mathbf{H}_s'\boldsymbol{\varepsilon},$$

so with a similar use of Lemma 2 we have

$$\text{(B5)}\quad \overline{\boldsymbol{\beta}}_s - (\mathbf{H}_s'\mathbf{H}_s)^{-1}\mathbf{H}_s'\mathbf{H}\boldsymbol{\beta} \overset{p}{\to} 0.$$

Since $\mathbf{H}_s'\mathbf{H}_s = \mathbf{H}_{ss}'\mathbf{H}_{ss} + \mathbf{H}_{sc}'\mathbf{H}_{sc}$ and $\mathbf{H}_s'\mathbf{H}\boldsymbol{\beta} = \mathbf{H}_{ss}'\mathbf{H}_{ss}\boldsymbol{\beta}_s + \mathbf{H}_{ss}'\mathbf{H}_{\sim ss}\boldsymbol{\beta}_{\sim s} + \mathbf{H}_{sc}'\mathbf{H}_{\bullet c}\boldsymbol{\beta}$, we have

$$\text{(B6)}\quad \overline{\boldsymbol{\beta}}_s \overset{p}{\to} \boldsymbol{\beta}_s + (\mathbf{H}_{ss}'\mathbf{H}_{ss} + \mathbf{H}_{sc}'\mathbf{H}_{sc})^{-1}[-\mathbf{H}_{sc}'\mathbf{H}_{sc}\boldsymbol{\beta}_s + \mathbf{H}_{ss}'\mathbf{H}_{\sim ss}\boldsymbol{\beta}_{\sim s} + \mathbf{H}_{sc}'\mathbf{H}_{\bullet c}\boldsymbol{\beta}].$$

We now consider the possibility that the limit of t_s/t_c is infinite along a particular equilibrium path. As in these circumstances the limit of t_s must be infinite, we can calculate the following probability limits

$$\text{(B7)}\quad \frac{\mathbf{H}_{ss}'\mathbf{H}_{\sim ss}}{t_sR} = \frac{\mathbf{X}_{ss}'\mathbf{N}_{\sim ss}}{t_sR} + \frac{\mathbf{N}_{ss}'\mathbf{N}_{\sim ss}}{t_sR} \overset{p}{\to} \mathbf{0}_{k_s x k_{\sim s}}$$

$$\frac{\mathbf{H}_{ss}'\mathbf{H}_{ss}}{t_sR} - \frac{\mathbf{X}_{ss}'\mathbf{X}_{ss}}{t_sR} = \frac{\mathbf{N}_{ss}'\mathbf{X}_{ss}}{t_sR} + \frac{\mathbf{X}_{ss}'\mathbf{N}_{ss}}{t_sR} + \frac{\mathbf{N}_{ss}'\mathbf{N}_{ss}}{t_sR} \overset{p}{\to} \frac{\sigma_n^2}{R}\mathbf{I}_{k_s},$$

$$\&\text{ if } \lim t_c < \infty,\quad \frac{\mathbf{H}_{sc}'\mathbf{H}_{sc} - \mathbf{X}_{sc}'\mathbf{X}_{sc}}{t_sR} \overset{p}{\to} \mathbf{0}_{k_s x k_s},\quad \frac{\mathbf{H}_{sc}'\mathbf{H}_{sc}}{t_sR} \overset{p}{\to} \mathbf{0}_{k_s x k_s},\ \&\ \frac{\mathbf{H}_{sc}'\mathbf{H}_{\bullet c}}{t_sR} \overset{p}{\to} \mathbf{0}_{k_s x k},$$

$$\text{or if } \lim t_c = \infty,\quad \frac{\mathbf{H}_{sc}'\mathbf{H}_{sc} - \mathbf{X}_{sc}'\mathbf{X}_{sc}}{t_sR} = \frac{t_c}{t_s}\left[\frac{\mathbf{N}_{sc}'\mathbf{X}_{sc}}{t_cR} + \frac{\mathbf{X}_{sc}'\mathbf{N}_{sc}}{t_cR} + \frac{\mathbf{N}_{sc}'\mathbf{N}_{sc}}{t_cR}\right] \overset{p}{\to} \frac{1}{\infty}\frac{\sigma_n^2}{R}\mathbf{I}_{k_s} = \mathbf{0}_{k_s x k_s}$$

$$\frac{\mathbf{H}_{sc}'\mathbf{H}_{sc}}{t_sR} = \frac{t_c}{t_s}\left[\frac{\mathbf{X}_{sc}'\mathbf{X}_{sc}}{t_cR} + \frac{\mathbf{N}_{sc}'\mathbf{X}_{sc}}{t_cR} + \frac{\mathbf{X}_{sc}'\mathbf{N}_{sc}}{t_cR} + \frac{\mathbf{N}_{sc}'\mathbf{N}_{sc}}{t_cR}\right] \overset{p}{\to} \frac{1}{\infty}\left[\frac{\boldsymbol{\beta}_s\boldsymbol{\beta}_s'}{\boldsymbol{\beta}'\boldsymbol{\beta}} + \frac{\sigma_n^2}{R}\mathbf{I}_{k_s}\right] = \mathbf{0}_{k_s x k_s}$$

$$\&\ \frac{\mathbf{H}_{sc}'\mathbf{H}_{\bullet c}}{t_sR} = \frac{t_c}{t_s}\left[\frac{\mathbf{X}_{sc}'\mathbf{X}_{\bullet c}}{t_cR} + \frac{\mathbf{N}_{sc}'\mathbf{X}_{\bullet c}}{t_cR} + \frac{\mathbf{X}_{sc}'\mathbf{N}_{\bullet c}}{t_cR} + \frac{\mathbf{N}_{sc}'\mathbf{N}_{\bullet c}}{t_cR}\right] \overset{p}{\to} \frac{1}{\infty}\left[\frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'}{\boldsymbol{\beta}'\boldsymbol{\beta}} + [\frac{\sigma_n^2}{R}\mathbf{I}_{k_s}, \mathbf{0}_{k_s x k_{\sim s}}]\right] = \mathbf{0}_{k_s x k}$$

where we make use of Lemma 1 earlier, the fact that $\mathbf{H}_{\sim ss} = \mathbf{N}_{\sim ss}$, as the simple set all policies they believe are irrelevant to zero, and in the last four lines that either the limit of t_c is finite, in which case we are dividing the sum of a finite number of random variables by a number (t_s) that goes to infinity, or the limit of t_c is infinite, in which we are dividing matrices that have finite probability limits by a number (t_s/t_c) that goes to infinity. Following the approach of the proof of Lemma 2 earlier, we can then argue that:

$$\text{(B8)}\quad \mathbf{v}'(\mathbf{H}_s'\mathbf{H}_s)^{-2}\mathbf{v} \le \frac{\mathbf{v}'\mathbf{v}}{\lambda_{\min}(\mathbf{H}_s'\mathbf{H}_s)^2} \le \frac{\mathbf{v}'\mathbf{v}/(t_sR)^2}{\lambda_{\min}\left((\mathbf{H}_s'\mathbf{H}_s - \mathbf{X}_s'\mathbf{X}_s)/t_sR\right)^2} \overset{p}{\to} \frac{\mathbf{0}_{k_s x1}'\mathbf{0}_{k_s x1}}{(\sigma_n^2/R)^2} = 0,$$

$$\Rightarrow \mathbf{v}'(\mathbf{H}_s'\mathbf{H}_s)^{-2}\mathbf{v} \overset{p}{\to} 0,\ \text{ where } \frac{\mathbf{v}}{t_sR} = -\frac{\mathbf{H}_{sc}'\mathbf{H}_{sc}}{t_sR}\boldsymbol{\beta}_s + \frac{\mathbf{H}_{ss}'\mathbf{H}_{\sim ss}}{t_sR}\boldsymbol{\beta}_{\sim s} + \frac{\mathbf{H}_{sc}'\mathbf{H}_{\bullet c}}{t_sR}\boldsymbol{\beta} \overset{p}{\to} \mathbf{0}_{k_s x1}.$$

Combined with (B6), this implies that simple beliefs converge on the true parameter values, i.e. $\overline{\boldsymbol{\beta}}_s \overset{p}{\to} \boldsymbol{\beta}_s$. In this case, however, the simple have strictly lower intensity than the complex and hence must lose power to the complex. In sum, if the limit of t_s/t_c is infinite, with a probability asymptotically approaching one the complex are always in power. Consequently, with the exception of equilibrium paths of probability measure zero, the limit of t_s/t_c can not in fact be infinite.[1] Going forward, we focus on equilibrium paths along which the limit of t_s/t_c is finite, which implies that the limit of t_c is infinite.

We now consider the possibility that the simple are in power only a finite number of times. In this case, as the complex will be in power an infinite number of times, we use Lemma 1 again to calculate

$$\text{(B9)}\quad \frac{\mathbf{H}'_{sc}\mathbf{H}_{sc}}{t_c R} = \frac{\mathbf{X}'_{sc}\mathbf{X}_{sc}}{t_c R} + \frac{\mathbf{N}'_{sc}\mathbf{X}_{sc}}{t_c R} + \frac{\mathbf{X}'_{sc}\mathbf{N}_{sc}}{t_c R} + \frac{\mathbf{N}'_{sc}\mathbf{N}_{sc}}{t_c R} \xrightarrow{p} \frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'_s}{\boldsymbol{\beta}'\boldsymbol{\beta}} + \frac{\sigma_n^2}{R}\mathbf{I}_{k_s}$$

$$\frac{\mathbf{H}'_{sc}\mathbf{H}_{\bullet c}}{t_c R} = \frac{\mathbf{X}'_{sc}\mathbf{X}_{\bullet c}}{t_c R} + \frac{\mathbf{N}'_{sc}\mathbf{X}_{\bullet c}}{t_c R} + \frac{\mathbf{X}'_{sc}\mathbf{N}_{\bullet c}}{t_c R} + \frac{\mathbf{N}'_{sc}\mathbf{N}_{\bullet c}}{t_c R} \xrightarrow{p} \frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'}{\boldsymbol{\beta}'\boldsymbol{\beta}} + \frac{\sigma_n^2}{R}[\mathbf{I}_{k_s}, \mathbf{0}_{k_s x k_{\sim s}}]$$

$$\& \quad \frac{\mathbf{H}'_{ss}\mathbf{H}_{ss}}{t_c R} \xrightarrow{p} \mathbf{0}_{k_s x k_s}, \quad \frac{\mathbf{H}'_{ss}\mathbf{H}_{\sim ss}}{t_c R} \xrightarrow{p} \mathbf{0}_{k_s x k_{\sim s}}$$

Applying these to (B6), we then conclude that if the limit of t_s is finite

$$\text{(B10)}\quad \bar{\boldsymbol{\beta}}_s \xrightarrow{p} \boldsymbol{\beta}_s + \left[\frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'_s}{\boldsymbol{\beta}'\boldsymbol{\beta}} + \frac{\sigma_n^2}{R}\mathbf{I}_{k_s}\right]^{-1}\left[-\left[\frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'_s}{\boldsymbol{\beta}'\boldsymbol{\beta}} + \frac{\sigma_n^2}{R}\mathbf{I}_{k_s}\right]\boldsymbol{\beta}_s + \frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'}{\boldsymbol{\beta}'\boldsymbol{\beta}}\boldsymbol{\beta} + \frac{\sigma_n^2}{R}[\mathbf{I}_{k_s}, \mathbf{0}_{k_s x k_{\sim s}}]\boldsymbol{\beta}\right]$$

$$= \boldsymbol{\beta}_s + \left[(R/\sigma_n^2)\mathbf{I}_{k_s} - \frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'_s(R/\sigma_n^2)^2/\boldsymbol{\beta}'\boldsymbol{\beta}}{1+\boldsymbol{\beta}'_s\boldsymbol{\beta}_s(R/\sigma_n^2)/\boldsymbol{\beta}'\boldsymbol{\beta}}\right]\left(1 - \frac{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}{\boldsymbol{\beta}'\boldsymbol{\beta}}\right)\boldsymbol{\beta}_s = \boldsymbol{\beta}_s + \frac{(1-\boldsymbol{\beta}'_s\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta})}{\sigma_n^2/R + \boldsymbol{\beta}'_s\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta}}\boldsymbol{\beta}_s.$$

From (B10), we see that as the ratio of noise to the information revealed by policy (σ_n^2/R) goes to infinity, $\bar{\boldsymbol{\beta}}_s \xrightarrow{p} \boldsymbol{\beta}_s$. This implies that asymptotically the simple have strictly lower voting intensity than the complex, which is consistent with their being in power only a finite number of times. In contrast, as σ_n^2/R goes to 0, (B10) reduces to $\bar{\boldsymbol{\beta}}_s \xrightarrow{p} \boldsymbol{\beta}_s(\boldsymbol{\beta}'\boldsymbol{\beta}/\boldsymbol{\beta}'_s\boldsymbol{\beta}_s)$, which implies that asymptotically with a probability approaching one the simple's voting intensity is greater than that of the complex, thereby, with the exception of equilibrium paths of probability measure zero, contradicting the assumption

[1]Since in the probability limit simple intensity is less than that of the complex, any paths such that the frequency in any fixed time interval that the simple are in power is asymptotically bounded above zero (as is necessary for the limit of t_s/t_c to go to infinity) must be of zero measure in probability.

that the simple are only in power a finite number of times.[2] Going forward, we shall assume σ_n^2 / R is sufficiently small to ensure this is the case. Along with our earlier results, this implies that, outside of a set of equilibrium paths of probability measure zero, along any other equilibrium path the limits of both t_c and t_s are infinite, while the limit of t_s/t_c is finite. We focus on such paths.

With the preceding in hand, we can conclude

$$\text{(B11)}\quad \frac{\mathbf{H}'_{sc}\mathbf{H}_{sc}}{t_c R} \xrightarrow{p} \frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'_s}{\boldsymbol{\beta}'\boldsymbol{\beta}} + \frac{\sigma_n^2}{R}\mathbf{I}_{k_s} \;\&\; \frac{\mathbf{H}'_{sc}\mathbf{H}_{\bullet c}}{t_c R} \xrightarrow{p} \frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'}{\boldsymbol{\beta}'\boldsymbol{\beta}} + \frac{\sigma_n^2}{R}[\mathbf{I}_{k_s}, \mathbf{0}_{k_s x k_{\sim s}}] \text{ (both as in (B9)}$$

$$\& \quad \frac{\mathbf{H}'_{ss}\mathbf{H}_{ss} - \mathbf{X}'_{ss}\mathbf{X}_{ss}}{t_s R} = \frac{\mathbf{N}'_{ss}\mathbf{X}_{ss}}{t_s R} + \frac{\mathbf{X}'_{ss}\mathbf{N}_{ss}}{t_s R} + \frac{\mathbf{N}'_{ss}\mathbf{N}_{ss}}{t_s R} \xrightarrow{p} \frac{\sigma_n^2}{R}\mathbf{I}_{k_s} \text{ and } \frac{\mathbf{H}'_{ss}\mathbf{H}_{\sim ss}}{t_s R} \xrightarrow{p} \mathbf{0}_{k_s x k_{\sim s}}.$$

Applying these to (B6) we see that

$$\text{(B12)}\quad \bar{\boldsymbol{\beta}}_s \xrightarrow{p} \boldsymbol{\beta}_s + \left[\frac{\mathbf{X}'_{ss}\mathbf{X}_{ss}}{t_c R} + \frac{t_s}{t_c}\frac{\sigma_n^2}{R}\mathbf{I}_{k_s} + \frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'_s}{\boldsymbol{\beta}'\boldsymbol{\beta}} + \frac{\sigma_n^2}{R}\mathbf{I}_{k_s}\right]^{-1} *$$

$$\left(-\left[\frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'_s}{\boldsymbol{\beta}'\boldsymbol{\beta}} + \frac{\sigma_n^2}{R}\mathbf{I}_{k_s}\right]\boldsymbol{\beta}_s + \left[\frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'}{\boldsymbol{\beta}'\boldsymbol{\beta}} + \frac{\sigma_n^2}{R}[\mathbf{I}_{k_s}, \mathbf{0}_{k_s x k_{\sim s}}]\right]\boldsymbol{\beta}\right)$$

$$\rightarrow \quad \bar{\boldsymbol{\beta}}_s \xrightarrow{p} \boldsymbol{\beta}_s + c\mathbf{M}^{-1}\boldsymbol{\beta}_s, \text{ where } \mathbf{M} = \frac{\mathbf{X}'_{ss}\mathbf{X}_{ss}}{t_c R} + \frac{t_c + t_s}{t_c}\frac{\sigma_n^2}{R}\mathbf{I}_{k_s} + \frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'_s}{\boldsymbol{\beta}'\boldsymbol{\beta}} \quad \& \quad c = 1 - \frac{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}{\boldsymbol{\beta}'\boldsymbol{\beta}}.$$

We note that

$$\text{(B13)}\quad \lambda_{\max}(\mathbf{M}^{-1}) = \lambda_{\min}(\mathbf{M})^{-1} \le \lambda_{\min}\left(\mathbf{M} - \frac{\mathbf{X}'_{ss}\mathbf{X}_{ss}}{t_c R}\right)^{-1}$$

$$= \lambda_{\min}\left(\frac{t_c + t_s}{t_c}\frac{\sigma_n^2}{R}\mathbf{I}_s + \frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'_s}{\boldsymbol{\beta}'\boldsymbol{\beta}}\right)^{-1} = \left(\frac{t_c + t_s}{t_c}\frac{\sigma_n^2}{R}\right)^{-1} \le \frac{R}{\sigma_n^2}$$

so the product of $\mathbf{M}^{-1}$ times the probability limits of $\mathbf{H}'_{sc}\mathbf{H}_{sc} / t_c R$ and $\mathbf{H}'_{sc}\mathbf{H}_{\bullet c} / t_c R$, and $\mathbf{M}^{-1} * t_s/t_c$ times the plim of $\mathbf{H}'_{ss}\mathbf{H}_{\sim ss} / t_s R$, all as given in (B11), is bounded, thereby validating the transition from (B6) to (B12).

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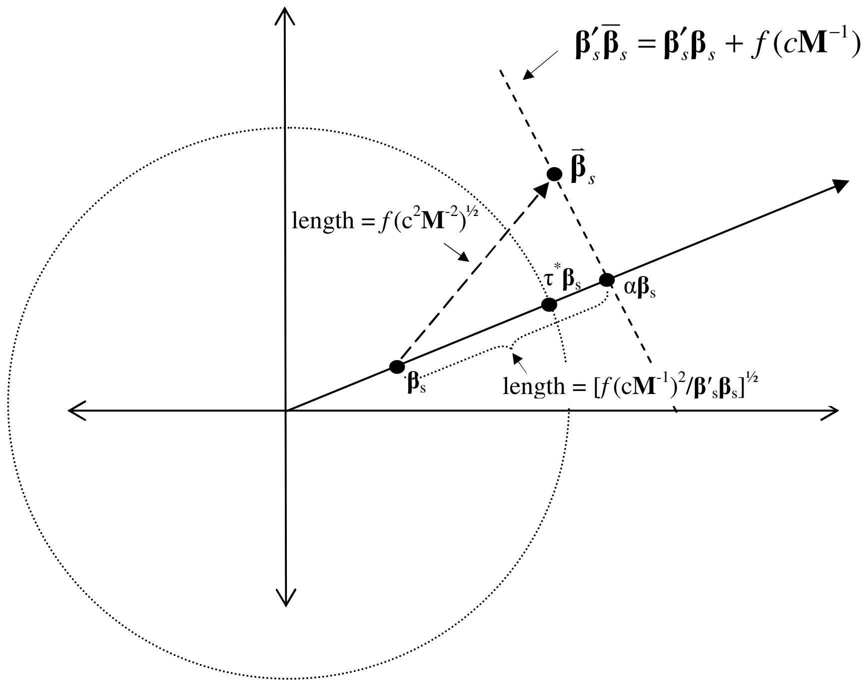
[2]Similar to the previous case, if the probability limit of simple intensity is greater than that of the complex, then paths such that the frequency in any fixed time interval that the complex are in power asymptotically goes to 1 (necessary for the limit of t_s to be finite) must be of zero measure in probability.

Figure B1: $f(c\mathbf{M}^{-1})$ and $f(c^2\mathbf{M}^{-2})$ for Two-Dimensional Simple



From (B12), we see that the asymptotic intensity of the simple equals

$$\text{(B14)}\quad \bar{\boldsymbol\beta}_s'\bar{\boldsymbol\beta}_s \xrightarrow{p} \boldsymbol\beta_s'\boldsymbol\beta_s + 2f(c\mathbf{M}^{-1}) + f(c^2\mathbf{M}^{-2}),$$

where $f(c\mathbf{M}^{-1}) = c\boldsymbol\beta_s'\mathbf{M}^{-1}\boldsymbol\beta_s$ and $f(c^2\mathbf{M}^{-2}) = c^2\boldsymbol\beta_s'\mathbf{M}^{-1}\mathbf{M}^{-1}\boldsymbol\beta_s$

are quadratic forms involving $\boldsymbol\beta_s$. Moreover, from (B12) we also see that

$$\text{(B15a)}\quad \boldsymbol\beta_s'(\bar{\boldsymbol\beta}_s - \boldsymbol\beta_s) \xrightarrow{p} f(c\mathbf{M}^{-1}) \quad \& \quad \text{(B15b)}\quad (\bar{\boldsymbol\beta}_s - \boldsymbol\beta_s)'(\bar{\boldsymbol\beta}_s - \boldsymbol\beta_s) \xrightarrow{p} f(c^2\mathbf{M}^{-2}),$$

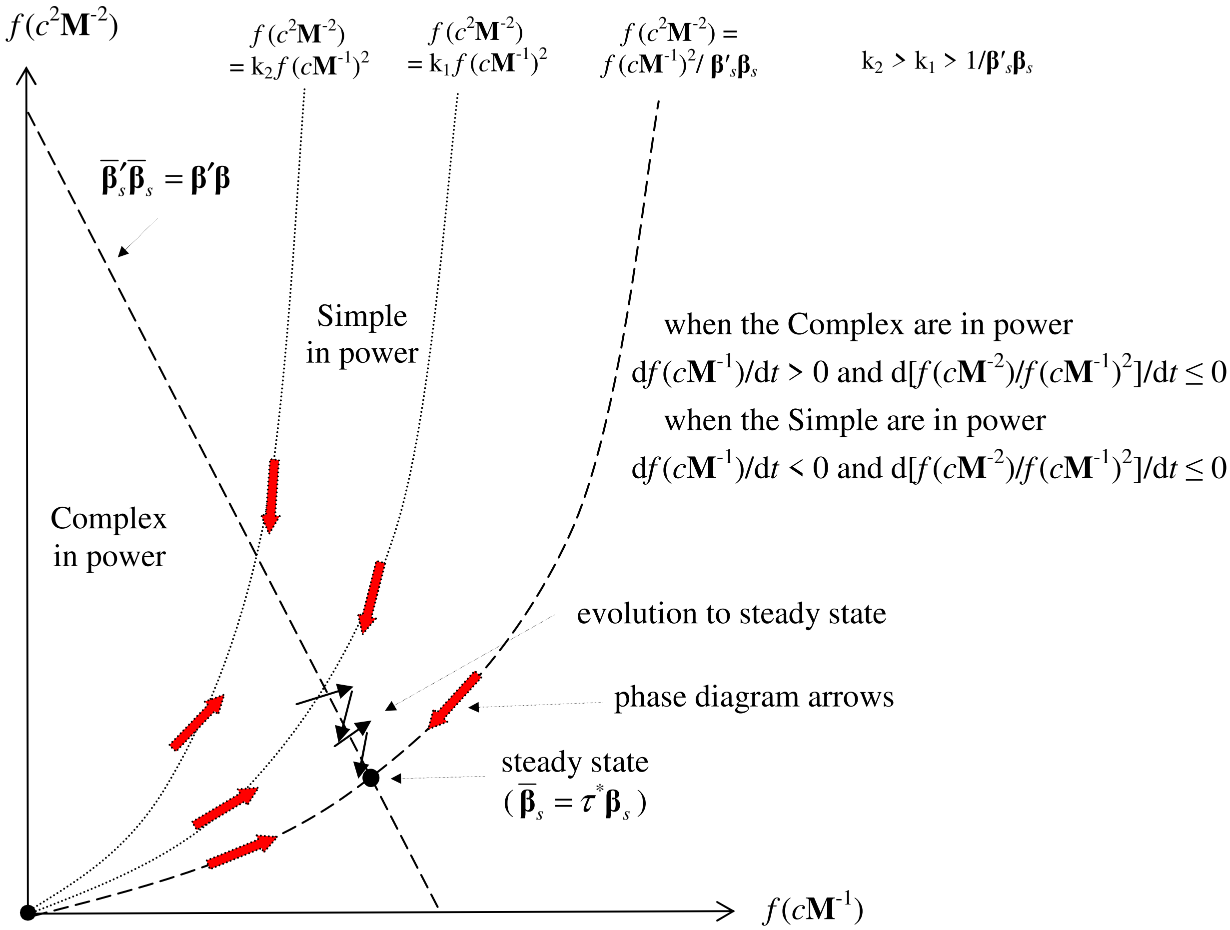
so $f(c\mathbf{M}^{-1})$ is the equation of a plane perpendicular to the ray from the origin defined by the true parameter values, while $f(c^2\mathbf{M}^{-2})^{½}$ is the distance of the ray from the true parameter values to mean beliefs. These are illustrated graphically, for the case where $\boldsymbol\beta_s$ involves two policies, in Figure B1 below.

If $\alpha\boldsymbol\beta_s$ denotes the coordinates of the intersection of the plane defined by (B15a) with the ray from the origin defined by $\boldsymbol\beta_s$ (see Fig B1), we can substitute $\alpha\boldsymbol\beta_s$ for $\bar{\boldsymbol\beta}_s$ in (B15a)

$$\text{(B16)}\quad \boldsymbol\beta_s'(\alpha\boldsymbol\beta_s - \boldsymbol\beta_s) \xrightarrow{p} f(c\mathbf{M}^{-1}) \Rightarrow (\alpha - 1)^2\boldsymbol\beta_s'\boldsymbol\beta_s \xrightarrow{p} f(c\mathbf{M}^{-1})^2 / \boldsymbol\beta_s'\boldsymbol\beta_s$$

However, the square of the length of the line segment from $\boldsymbol\beta_s$ to $\alpha\boldsymbol\beta_s$ is also $(\alpha - 1)^2\boldsymbol\beta_s'\boldsymbol\beta_s$. By the Pythagorean theorem this must be less than or equal to the square of the length of

Figure B2: Asymptotic Phase Diagram for $f(c^2\mathbf{M}^{-2})$ and $f(c\mathbf{M}^{-1})$



the line segment from $\boldsymbol{\beta}_s$ to $\bar{\boldsymbol{\beta}}_s$, which equals $f(c^2\mathbf{M}^{-2})$. Consequently, $f(c^2\mathbf{M}^{-2}) \geq f(c\mathbf{M}^{-1})^2/\boldsymbol{\beta}'_s\boldsymbol{\beta}_s$, with equality only when $\bar{\boldsymbol{\beta}}_s$ actually equals $\alpha\boldsymbol{\beta}_s$.[3] In sum, another interpretation of $f(c\mathbf{M}^{-1})$ is that it is proportional to the projection of the deviation of the simple's beliefs from the truth ($\boldsymbol{\beta}_s$) on the direction $\boldsymbol{\beta}_s$, a measure of bias, while the ratio $f(c^2\mathbf{M}^{-2})/[f(c\mathbf{M}^{-1})^2/\boldsymbol{\beta}'_s\boldsymbol{\beta}_s]$ is the secant^2 of the angle of deviation from the direction $\boldsymbol{\beta}_s$.

Figure B2 draws the asymptotic phase diagram for $f(c\mathbf{M}^{-1})$ and $f(c^2\mathbf{M}^{-2})$. The downward sloping dashed line, with slope -2, denotes the combinations that are consistent with $\bar{\boldsymbol{\beta}}'_s\bar{\boldsymbol{\beta}}_s = \boldsymbol{\beta}'\boldsymbol{\beta}$, i.e. the simple having the same voting intensity as the

[3]This result is also an implication of the generalized Cauchy-Schwarz inequality, which states that for a positive definite matrix $\mathbf{S}$, and vectors $\mathbf{x}$ and $\mathbf{y}$, $(\mathbf{x}'\mathbf{y})^2 \leq \mathbf{x}'\mathbf{S}\mathbf{y}*\mathbf{x}'\mathbf{S}^{-1}\mathbf{y}$ (Anderson 2003). Letting $\mathbf{x} = \mathbf{y} = c^{1/2}\boldsymbol{\beta}_s\mathbf{M}^{-1/2}$ and $\mathbf{S} = \mathbf{M}$, we have $(c\boldsymbol{\beta}'_s\mathbf{M}^{-1}\boldsymbol{\beta}_s)^2 \leq c\boldsymbol{\beta}'_s\boldsymbol{\beta}_s c\boldsymbol{\beta}'_s\mathbf{M}^{-1}\mathbf{M}^{-1}\boldsymbol{\beta}_s \rightarrow f(c^2\mathbf{M}^{-2}) \geq f(c\mathbf{M}^{-1})^2/\boldsymbol{\beta}'_s\boldsymbol{\beta}_s$.

complex, based on (B14) above. Above the line the simple are in power, while below the line the complex are in power. Also drawn in the figure are "level curves" of the form $f(c^2\mathbf{M}^{-2}) = k*f(c\mathbf{M}^{-1})^2$, with each curve defined by a different value of the constant k. The lowest curve, with $f(c^2\mathbf{M}^{-2}) = f(c^2\mathbf{M}^{-1})^2/\boldsymbol{\beta}'_s\boldsymbol{\beta}_s$, passes through the steady state, as $\overline{\boldsymbol{\beta}}_s$ there is proportional to $\boldsymbol{\beta}_s$. We prove the following results further below:

(B17a) If the complex are in power $\mathrm{d}f(c\mathbf{M}^{-1})/\mathrm{d}t > 0$ and $\mathrm{d}[f(c^2\mathbf{M}^{-2})/f(c\mathbf{M}^{-1})^2]/\mathrm{d}t \leq 0$, with equality only along the steady state level curve;

(B17b) If the simple are in power $\mathrm{d}f(c\mathbf{M}^{-1})/\mathrm{d}t < 0$ and $\mathrm{d}[f(c^2\mathbf{M}^{-2})/f(c\mathbf{M}^{-1})^2]/\mathrm{d}t \leq 0$, with equality only along the steady state level curve.

(B17c) No matter which type is in power, $\lim_{t\to\infty} \mathrm{d}f(c\mathbf{M}^{-1})/\mathrm{d}t = 0$.

Asymptotically, when the complex are in power, bias as measured by the projection onto the directional vector given by the truth monotonically increases ($\mathrm{d}f(c\mathbf{M}^{-1})/\mathrm{d}t > 0$), while when the simple are in power it monotonically declines ($\mathrm{d}f(c\mathbf{M}^{-1})/\mathrm{d}t < 0$). Regardless of which type is in power, the angle of the deviation of beliefs from the direction implied by true parameter values monotonically falls, $\mathrm{d}[f(c^2\mathbf{M}^{-2})/f(c\mathbf{M}^{-1})^2]/\mathrm{d}t \leq 0$. As shown formally below, this effect comes from two factors: (i) noise, which regardless of which type is in power lowers the directional deviation of beliefs from $\boldsymbol{\beta}_s$, and (ii) the policy actions of the simple which, insofar as they are not proportional to $\boldsymbol{\beta}_s$, when contrasted with the actions of the complex reveal information about the relative effects of the k_s policies the simple consider relevant. The asymptotic collinearity of complex actions means that the effects of policies the simple believe are irrelevant can be loaded upon on any of the policies they believe are relevant. The effects of this bias are expressed in the form of movements of the line defined by $\boldsymbol{\beta}'_s\overline{\boldsymbol{\beta}}_s = \boldsymbol{\beta}'_s\boldsymbol{\beta}_s + f(c\mathbf{M}^{-1})$, but simple beliefs in principle could lie anywhere on this line. It is noise, plus the contrast between the effects of simple and complex actions when simple policies are not collinear in the area of overlap, that gradually reduces the deviation along this line from the ray $\alpha\boldsymbol{\beta}_s$.

(B17a) and (B17b) together establish that in the probability limit simple beliefs evolve toward the steady state following zig-zag paths such as the one drawn in the figure. (B17c), along with the monotonicity of $f(c^2\mathbf{M}^{-2})/f(c\mathbf{M}^{-1})^2$, ensures that these movements eventually stop. As a final step, we need to show that when simple beliefs stop moving they must be at the steady state given in the figure, i.e. they cannot converge on some earlier point in the phase diagram path. We will first show that if simple beliefs converge they must converge to a point on the lowest level curve of the phase diagram, where simple beliefs are proportional to $\boldsymbol{\beta}_s$, and then show that this implies convergence to the steady state.

We return to the equation $\overline{\boldsymbol{\beta}}_s = \boldsymbol{\beta}_s + c\mathbf{M}^{-1}\boldsymbol{\beta}_s$, as defined in (B12), plugging in the probability limit of $\mathbf{X}'_{ss}\mathbf{X}_{ss}/t_s$ given knowledge that simple beliefs converge

$$\text{(B18)}\ \ \mathbf{M} = \mathbf{V} + \frac{\boldsymbol{\beta}_s\boldsymbol{\beta}_s'}{\boldsymbol{\beta}'\boldsymbol{\beta}} \ \text{ with } \ \mathbf{V} = \frac{t_s}{t_c}\frac{\overline{\boldsymbol{\beta}}_s\overline{\boldsymbol{\beta}}_s'}{\overline{\boldsymbol{\beta}}_s'\overline{\boldsymbol{\beta}}_s} + \frac{t_c + t_s}{t_c}\frac{\sigma_n^2}{R}\mathbf{I}_{k_s},$$

$$\text{so}\ \ \mathbf{M}^{-1}\boldsymbol{\beta}_s = \left[\mathbf{V}^{-1} - \frac{\mathbf{V}^{-1}\boldsymbol{\beta}_s\boldsymbol{\beta}_s'\mathbf{V}^{-1}/\boldsymbol{\beta}'\boldsymbol{\beta}}{1+\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta}}\right]\boldsymbol{\beta}_s = \frac{\mathbf{V}^{-1}\boldsymbol{\beta}_s}{1+\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta}}$$

$$\text{and}\ \mathbf{V}^{-1} = t_c\left[\frac{R}{t\sigma_n^2}\mathbf{I}_{k_s} - \frac{t_s\overline{\boldsymbol{\beta}}_s\overline{\boldsymbol{\beta}}_s'(R/t\sigma_n^2)^2/\overline{\boldsymbol{\beta}}_s'\overline{\boldsymbol{\beta}}_s}{1+t_s(R/t\sigma_n^2)}\right]$$

$$\Rightarrow\ \ \overline{\boldsymbol{\beta}}_s \xrightarrow{p} \boldsymbol{\beta}_s + \frac{ct_c}{1+\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta}}\left[\frac{R}{t\sigma_n^2}\boldsymbol{\beta}_s - \frac{t_s\overline{\boldsymbol{\beta}}_s'\boldsymbol{\beta}_s R^2/t^2\sigma_n^4\overline{\boldsymbol{\beta}}_s'\overline{\boldsymbol{\beta}}_s}{1+t_sR/t\sigma_n^2}\overline{\boldsymbol{\beta}}_s\right],$$

$$\Rightarrow\ \ \overline{\boldsymbol{\beta}}_s \xrightarrow{p} \tau\boldsymbol{\beta}_s,\ \text{ where }\ \tau = \frac{1+\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta} + c(R/\sigma_n^2)(t_c/t)}{1+\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta} + \dfrac{ct_ct_s\overline{\boldsymbol{\beta}}_s'\boldsymbol{\beta}_s R^2/t^2\sigma_n^4\overline{\boldsymbol{\beta}}_s'\overline{\boldsymbol{\beta}}_s}{1+t_sR/t\sigma_n^2}},$$

so we see that simple beliefs in the probability limit must be proportional to $\boldsymbol{\beta}_s$. We use this fact to calculate to $\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s'$ and substitute in the expression for τ (using as well the fact that $\overline{\boldsymbol{\beta}}_s'\boldsymbol{\beta}_s/\overline{\boldsymbol{\beta}}_s'\overline{\boldsymbol{\beta}}_s = 1/\tau$)

$$\text{(B19)}\ \ \boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s = t_c\left[\frac{R\boldsymbol{\beta}_s'\boldsymbol{\beta}_s}{t\sigma_n^2} - \frac{t_s\boldsymbol{\beta}_s'\boldsymbol{\beta}_sR^2/t^2\sigma_n^4}{1+t_sR/t\sigma_n^2}\right] = \frac{t_cR\boldsymbol{\beta}_s'\boldsymbol{\beta}_s/t\sigma_n^2}{1+t_sR/t\sigma_n^2}$$

$$\Rightarrow\ \ \tau = \frac{1+\dfrac{\boldsymbol{\beta}_s'\boldsymbol{\beta}_s}{\boldsymbol{\beta}'\boldsymbol{\beta}}\dfrac{t_cR/t\sigma_n^2}{1+t_sR/t\sigma_n^2} + c(R/\sigma_n^2)(t_c/t)}{1+\dfrac{\boldsymbol{\beta}_s'\boldsymbol{\beta}_s}{\boldsymbol{\beta}'\boldsymbol{\beta}}\dfrac{t_cR/t\sigma_n^2}{1+t_sR/t\sigma_n^2} + \dfrac{1}{\tau}\dfrac{ct_ct_sR^2/t^2\sigma_n^4}{1+t_sR/t\sigma_n^2}}$$

$$\Rightarrow\ \ \tau = 1 + \frac{c(R/\sigma_n^2)(t_c/t)}{1+(t_s/t)(R/\sigma_n^2) + \dfrac{\boldsymbol{\beta}_s'\boldsymbol{\beta}_s}{\boldsymbol{\beta}'\boldsymbol{\beta}}(t_c/t)(R/\sigma_n^2)}$$

The right hand side of the last line is decreasing in t_s/t and increasing in t_c/t, so we have

$$\text{(B20)}\quad \frac{d\tau}{d(t_s/t)}<0,\ \ \frac{d\tau}{d(t_c/t)}>0,\ \ \lim_{\frac{t_c}{t}\to 0,\frac{t_s}{t}\to 1}\tau=1\ \ \&\ \ \lim_{\frac{t_c}{t}\to 1,\frac{t_s}{t}\to 0}\tau=1+\frac{(1-\boldsymbol{\beta}_s'\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta})}{\sigma_n^2/R+\boldsymbol{\beta}_s'\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta}}.$$

The last expression was encountered earlier in (B10) and as σ_n^2/R goes to zero leads to a bias level $\tau=\tau^{*2}>\tau^*$.

(B18) - (B20) together ensure that movement in Figure B2 continues until simple beliefs converge on the steady state with bias τ equal to τ^*. In the probability limit beliefs must be proportional to $\boldsymbol{\beta}_s$. When $t_s/t = 1 - t_c/t$ is such that in the probability limit bias is greater than τ^*, with a probability approaching one the simple will be in power and t_s/t will rise while t_c/t falls, ensuring that τ falls, with opposite effects when τ is less than τ^* and the complex are in power. For small enough σ_n^2/R the limiting values of τ as t_s/t goes to zero and one encompass τ^*, ensuring that the probability limit of t_s/t is the one consistent with bias equal to the steady state value τ^*, as given in the text.

We now prove (B17a) and (B17b), turning to (B17c) at the end. We start by calculating expressions for $f(c\mathbf{M}^{-1})$ and $f(c^2\mathbf{M}^{-2})$ using (B12) and the Sherman-Morrison formula:

$$\text{(B21)}\quad f(c\mathbf{M}^{-1})=c\boldsymbol{\beta}_s'\left[\mathbf{V}+\frac{\boldsymbol{\beta}_s\boldsymbol{\beta}_s'}{\boldsymbol{\beta}'\boldsymbol{\beta}}\right]^{-1}\boldsymbol{\beta}_s,\ \text{ where }\ \mathbf{V}=\frac{\mathbf{X}_{ss}'\mathbf{X}_{ss}}{t_cR}+\frac{t_s+t_c}{t_c}\frac{\sigma_n^2}{R}\mathbf{I}_{k_s}$$

$$=c\boldsymbol{\beta}_s'\left[\mathbf{V}^{-1}-\frac{\mathbf{V}^{-1}\boldsymbol{\beta}_s\boldsymbol{\beta}_s'\mathbf{V}^{-1}/\boldsymbol{\beta}'\boldsymbol{\beta}}{1+\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta}}\right]\boldsymbol{\beta}_s\quad\Rightarrow\quad f(c\mathbf{M}^{-1})=\frac{c\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s}{1+\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta}}$$

$$f(c^2\mathbf{M}^{-2})=c^2\boldsymbol{\beta}_s'\left[\mathbf{V}+\frac{\boldsymbol{\beta}_s'\boldsymbol{\beta}_s}{\boldsymbol{\beta}'\boldsymbol{\beta}}\right]^{-1}\left[\mathbf{V}+\frac{\boldsymbol{\beta}_s'\boldsymbol{\beta}_s}{\boldsymbol{\beta}'\boldsymbol{\beta}}\right]^{-1}\boldsymbol{\beta}_s$$

$$=c^2\boldsymbol{\beta}_s'\left[\mathbf{V}^{-1}-\frac{\mathbf{V}^{-1}\boldsymbol{\beta}_s\boldsymbol{\beta}_s'\mathbf{V}^{-1}/\boldsymbol{\beta}'\boldsymbol{\beta}}{1+\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta}}\right]\left[\mathbf{V}^{-1}-\frac{\mathbf{V}^{-1}\boldsymbol{\beta}_s\boldsymbol{\beta}_s'\mathbf{V}^{-1}/\boldsymbol{\beta}'\boldsymbol{\beta}}{1+\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta}}\right]\boldsymbol{\beta}_s=$$

$$\Rightarrow\quad f(c^2\mathbf{M}^{-2})=\frac{c^2\boldsymbol{\beta}_s'\mathbf{V}^{-1}\mathbf{V}^{-1}\boldsymbol{\beta}_s}{(1+\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta})^2}\quad\Rightarrow\quad\frac{f(c^2\mathbf{M}^{-2})}{f(c\mathbf{M}^{-1})^2}=\frac{\boldsymbol{\beta}_s'\mathbf{V}^{-1}\mathbf{V}^{-1}\boldsymbol{\beta}_s}{(\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s)^2}$$

We then use the spectral decomposition of **V** to create two key expressions:

$$\text{(B22a)}\quad \boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s=\sum_{i=1}^{k_s}\lambda_i a_i^2,\quad\text{(B22b)}\quad\frac{\boldsymbol{\beta}_s'\mathbf{V}^{-1}\mathbf{V}^{-1}\boldsymbol{\beta}_s}{(\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s)^2}=\frac{\sum_{i=1}^{k_s}\lambda_i^2a_i^2}{\sum_{i=1}^{k_s}\lambda_ia_i^2\sum_{i=1}^{k_s}\lambda_ia_i^2}$$

where $\lambda_1 \geq ... \geq \lambda_i \geq ... \geq \lambda_{k_s}$ are the ordered eigenvalues of $\mathbf{V}^{-1}$ and the a_i the inner-products of the associated eigenvectors with $\boldsymbol{\beta}_s$, i.e. $\boldsymbol{a} = \mathbf{E}'\boldsymbol{\beta}_s$. From the matrix algebra results given earlier above, we know that:

$$\text{(B23)}\ \lambda_i = \frac{t_c R}{\gamma_i + (t_s + t_c)\sigma_n^2}$$

where $\gamma_1 \leq ... \leq \gamma_i \leq ... \leq \gamma_{k_s}$ are the ordered eigenvalues of $\mathbf{X}_{ss}'\mathbf{X}_{ss}$. While the λ_i are in descending order, the corresponding γ_i are in ascending order, as the two are inversely related. The eigenvector matrix $\mathbf{E}$ of $\mathbf{V}^{-1}$ is that of $\mathbf{X}_{ss}'\mathbf{X}_{ss}$ and hence, conditional on a given value of $\mathbf{X}_{ss}'\mathbf{X}_{ss}$, not a function of t_c, t_s or σ_n^2 / R. When beliefs are proportional to $\boldsymbol{\beta}_s$, only one of the a_i in (B22) is non-zero, i.e. one of the eigenvectors in $\mathbf{E}$ is $\boldsymbol{\beta}_s /(\boldsymbol{\beta}_s'\boldsymbol{\beta}_s)^{½}$ and the rest are orthogonal to $\boldsymbol{\beta}_s$. This can be seen by noting that

$$\text{(B24)}\ \alpha\boldsymbol{\beta}_s = \overline{\boldsymbol{\beta}}_s = \boldsymbol{\beta}_s + c\left[\mathbf{V} + \frac{\boldsymbol{\beta}_s\boldsymbol{\beta}_s'}{\boldsymbol{\beta}'\boldsymbol{\beta}}\right]^{-1}\boldsymbol{\beta}_s = \boldsymbol{\beta}_s + c\left[\mathbf{V}^{-1} - \frac{\mathbf{V}^{-1}\boldsymbol{\beta}_s\boldsymbol{\beta}_s'\mathbf{V}^{-1}/\boldsymbol{\beta}'\boldsymbol{\beta}}{1+\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta}}\right]\boldsymbol{\beta}_s$$

$$\Rightarrow \alpha\boldsymbol{\beta}_s = \boldsymbol{\beta}_s + \frac{c\mathbf{V}^{-1}\boldsymbol{\beta}_s}{1+\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta}} \quad \Rightarrow \quad \mathbf{V}^{-1}\boldsymbol{\beta}_s = \frac{(\alpha-1)(1+\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta})}{c}\boldsymbol{\beta}_s,$$

so $\boldsymbol{\beta}_s /(\boldsymbol{\beta}_s'\boldsymbol{\beta}_s)^{½}$ is an eigenvector of $\mathbf{V}^{-1}$.

When the complex are in power t_c is the only element that changes in $\mathbf{V}$ and hence the asymptotic effect on (B22a) and (B22b) can be calculated by simply looking at the implied changes in the eigenvalues in (B23). When the simple are in power, t_s changes, with effects through eigenvalues similar to those of the complex, but $\mathbf{X}_{ss}'\mathbf{X}_{ss}$ also changes, with effects on both the eigenvalues and eigenvectors, i.e. the a_i terms in (B23). We first calculate the effects of changes in t_c and t_s, and then examine the effects of changes in $\mathbf{X}_{ss}'\mathbf{X}_{ss}$, showing that they move (B22a) and (B22b) in the same direction as implied by increases in t_s.

Taking derivatives with respect to t_c and t_s, we have

$$\text{(B25)}\ \frac{d\lambda_i}{dt_c} = \frac{R(\gamma_i + t_s\sigma_n^2)}{(\gamma_i + (t_s+t_c)\sigma_n^2)^2} > 0 \ \text{ and } \ \frac{d\lambda_i}{dt_s} = -\frac{Rt_c\sigma_n^2}{(\gamma_i + (t_s+t_c)\sigma_n^2)^2} < 0.$$

From (B25) we see that when the complex are in power t_c increases and all of the eigenvalues of $\mathbf{V}^{-1}$ increase (with no change in the eigenvectors), so $\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s$ increases and, consequently, $f(c\mathbf{M}^{-1})$. When the simple are in power t_s increases, which lowers all of the eigenvalues of $\mathbf{V}^{-1}$ (without changing the eigenvectors) and hence lowers $f(c\mathbf{M}^{-1})$. Taking the derivative of (B22b) with respect to any eigenvalue, we find:

$$\text{(B26)}\quad \frac{\mathrm{d}\left(\dfrac{\boldsymbol{\beta}_s'\mathbf{V}^{-1}\mathbf{V}^{-1}\boldsymbol{\beta}_s}{(\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s)^2}\right)}{\mathrm{d}\lambda_i} = \frac{2a_i^2}{(\sum \lambda_i a_i^2)^3}\left[\lambda_i \sum_{j=1}^{k_s} \lambda_j a_j^2 - \sum_{j=1}^{k_s} \lambda_j^2 a_j^2\right]$$

So,

$$\text{(B27)}\quad \frac{\mathrm{d}\left(\dfrac{\boldsymbol{\beta}_s'\mathbf{V}^{-1}\mathbf{V}^{-1}\boldsymbol{\beta}_s}{(\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s)^2}\right)}{\mathrm{d}t_c} = \frac{2}{(\sum \lambda_i a_i^2)^3}\sum_{i=1}^{k_s} a_i^2\left[\lambda_i \sum_{j=1}^{k_s} \lambda_j a_j^2 - \sum_{j=1}^{k_s} \lambda_j^2 a_j^2\right]\frac{\mathrm{d}\lambda_i}{\mathrm{d}t_c}$$

$$= \frac{2}{(\sum \lambda_i a_i^2)^3}\sum_{i=1}^{k_s}\sum_{j=1}^{k_s} a_i^2 a_j^2(\lambda_i\lambda_j - \lambda_j^2)\frac{\mathrm{d}\lambda_i}{\mathrm{d}t_c}$$

$$= \frac{2}{(\sum \lambda_i a_i^2)^3}\sum_{i=1}^{k_s-1}\sum_{j=i+1}^{k_s} a_i^2 a_j^2\left[(\lambda_i\lambda_j - \lambda_j^2)\frac{\mathrm{d}\lambda_i}{\mathrm{d}t_c} + (\lambda_i\lambda_j - \lambda_i^2)\frac{\mathrm{d}\lambda_j}{\mathrm{d}t_c}\right] \le 0$$

as

$$\text{(B28)}\quad (\lambda_i\lambda_j - \lambda_j^2)\frac{\mathrm{d}\lambda_i}{\mathrm{d}t_c} + (\lambda_i\lambda_j - \lambda_i^2)\frac{\mathrm{d}\lambda_j}{\mathrm{d}t_c} = -\frac{\sigma_n^2(\gamma_i - \gamma_j)^2\lambda_i^3\lambda_j^3}{t_c^3 R^3} < 0,$$

with equality when $\sigma_n^2 = 0$ or a_i is non-zero for only one eigenvalue (i.e. the simple are on the level curve associated with the steady state with beliefs proportional to $\boldsymbol{\beta}_s$). Similarly,

$$\text{(B29)}\quad \frac{\mathrm{d}\left(\dfrac{\boldsymbol{\beta}_s'\mathbf{V}^{-1}\mathbf{V}^{-1}\boldsymbol{\beta}_s}{(\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s)^2}\right)}{\mathrm{d}t_s} = \frac{2}{(\sum \lambda_i a_i^2)^3}\sum_{i=1}^{k_s-1}\sum_{j=i+1}^{k_s} a_i^2 a_j^2\left[(\lambda_i\lambda_j - \lambda_j^2)\frac{\mathrm{d}\lambda_i}{\mathrm{d}t_s} + (\lambda_i\lambda_j - \lambda_i^2)\frac{\mathrm{d}\lambda_j}{\mathrm{d}t_s}\right] \le 0$$

as

$$\text{(B30)}\quad (\lambda_i\lambda_j - \lambda_j^2)\frac{\mathrm{d}\lambda_i}{\mathrm{d}t_s} + (\lambda_i\lambda_j - \lambda_i^2)\frac{\mathrm{d}\lambda_j}{\mathrm{d}t_s} = -\frac{\sigma_n^2(\gamma_i - \gamma_j)^2\lambda_i^3\lambda_j^3}{t_c^3 R^3} < 0,$$

with, once again equality when $\sigma_n^2 = 0$ or when beliefs are proportional to $\boldsymbol{\beta}_s$ and a_i is non-zero for only one eigenvalue. Intuition for why (B27) and (B29) are identical can be found by noting that while t_c appears in the numerator of (B23), this element implicitly cancels in the ratio (B22b). Consequently, all that is left is the influence of t_c and t_s in the denominator of (B23), where they are both multiplied by σ_n^2. As time passes, regardless of which type is in power, random noise lowers the angle of the deviation of the simple's beliefs from the direction implied by the true parameter values.

We now consider the impact of periods when the simple are in power through its effects on $\mathbf{X}'_{ss}\mathbf{X}_{ss}$. $f(c\mathbf{M}^{-1})$ is monotonically increasing in $\boldsymbol{\beta}'_s\mathbf{V}^{-1}\boldsymbol{\beta}_s$, with $\mathbf{V}$ as defined in (B21). Each period when the simple are in power and implement policies $\mathbf{x}$ generates a rank one update of $\mathbf{V}$, so that $\boldsymbol{\beta}'_s\mathbf{V}^{-1}\boldsymbol{\beta}_s$ becomes

$$\text{(B31)}\quad \boldsymbol{\beta}'_s\left[\mathbf{V}+\frac{\mathbf{xx}'}{t_cR}\right]^{-1}\boldsymbol{\beta}_s=\boldsymbol{\beta}'_s\left[\mathbf{V}^{-1}-\frac{\mathbf{V}^{-1}\mathbf{xx}'\mathbf{V}^{-1}/t_cR}{1+\mathbf{x}'\mathbf{V}^{-1}\mathbf{x}/t_cR}\right]\boldsymbol{\beta}_s$$

$$=\boldsymbol{\beta}'_s\mathbf{V}^{-1}\boldsymbol{\beta}_s-\frac{\boldsymbol{\beta}'_s\mathbf{V}^{-1}\mathbf{xx}'\mathbf{V}^{-1}\boldsymbol{\beta}_s/t_cR}{1+\mathbf{x}'\mathbf{V}^{-1}\mathbf{x}/t_cR}<\boldsymbol{\beta}'_s\mathbf{V}^{-1}\boldsymbol{\beta}_s,$$

so this effect lowers $f(c\mathbf{M}^{-1})$ as does (as already proven) the increase in t_s that accompanies periods when the simple are in power.

Turning to the ratio $f(c^2\mathbf{M}^{-2})/f(c\mathbf{M}^{-1})^2$, equal to $\boldsymbol{\beta}'_s\mathbf{V}^{-1}\mathbf{V}^{-1}\boldsymbol{\beta}_s/(\boldsymbol{\beta}'_s\mathbf{V}^{-1}\boldsymbol{\beta}_s)^2$ as shown in (B21), we again calculate the effects of the rank-one update of $\mathbf{V}$

$$\text{(B32)}\quad \frac{\boldsymbol{\beta}'_s\left[\mathbf{V}^{-1}-\frac{\mathbf{V}^{-1}\mathbf{xx}'\mathbf{V}^{-1}/t_cR}{1+\mathbf{x}'\mathbf{V}^{-1}\mathbf{x}/t_cR}\right]\left[\mathbf{V}^{-1}-\frac{\mathbf{V}^{-1}\mathbf{xx}'\mathbf{V}^{-1}/t_cR}{1+\mathbf{x}'\mathbf{V}^{-1}\mathbf{x}/t_cR}\right]\boldsymbol{\beta}_s}{\boldsymbol{\beta}'_s\left[\mathbf{V}^{-1}-\frac{\mathbf{V}^{-1}\mathbf{xx}'\mathbf{V}^{-1}/t_cR}{1+\mathbf{x}'\mathbf{V}^{-1}\mathbf{x}/t_cR}\right]\boldsymbol{\beta}_s\boldsymbol{\beta}'_s\left[\mathbf{V}^{-1}-\frac{\mathbf{V}^{-1}\mathbf{xx}'\mathbf{V}^{-1}/t_cR}{1+\mathbf{x}'\mathbf{V}^{-1}\mathbf{x}/t_cR}\right]\boldsymbol{\beta}_s}=$$

$$\frac{m^2_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}(1+m^1_{\mathbf{x}'\mathbf{x}}/t_cR)^2-2(m^2_{\boldsymbol{\beta}'_s\mathbf{x}}m^1_{\boldsymbol{\beta}'_s\mathbf{x}}/t_cR)(1+m^1_{\mathbf{x}'\mathbf{x}}/t_cR)+m^1_{\boldsymbol{\beta}'_s\mathbf{x}}m^2_{\mathbf{x}'\mathbf{x}}m^1_{\boldsymbol{\beta}'_s\mathbf{x}}/(t_cR)^2}{[m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}(1+m^1_{\mathbf{x}'\mathbf{x}}/t_cR)-m^1_{\boldsymbol{\beta}'_s\mathbf{x}}m^1_{\boldsymbol{\beta}'_s\mathbf{x}}/t_cR]^2},\text{ with } m^i_{\mathbf{a}'\mathbf{b}}=\mathbf{a}'\mathbf{V}^{-i}\mathbf{b}.$$

We wish to show this is $\leq \boldsymbol{\beta}'_s\mathbf{V}^{-1}\mathbf{V}^{-1}\boldsymbol{\beta}_s/(\boldsymbol{\beta}'_s\mathbf{V}^{-1}\boldsymbol{\beta}_s)^2=m^2_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}/m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}$, with equality only when $\overline{\boldsymbol{\beta}}_s$ is proportional to $\boldsymbol{\beta}_s$, i.e. when simple beliefs lie along the lowest level curve where $f(c^2\mathbf{M}^{-2})=f(c\mathbf{M}^{-1})^2/\boldsymbol{\beta}'_s\boldsymbol{\beta}_s$. If $\overline{\boldsymbol{\beta}}_s$ is proportional to $\boldsymbol{\beta}_s$, then so is policy implemented by the simple. Say $\mathbf{x}=\alpha\boldsymbol{\beta}_s$, then we have $m^i_{\boldsymbol{\beta}'_s\mathbf{x}}=\alpha m^i_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}$ and (B32) simplifies to:

$$\text{(B33)}\quad \frac{m^2_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}(1+m^1_{\mathbf{x}'\mathbf{x}}/t_cR)^2-2\alpha^2m^2_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}/t_cR)(1+m^1_{\mathbf{x}'\mathbf{x}}/t_cR)+\alpha^4m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}m^2_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}/(t_cR)^2}{[m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}(1+m^1_{\mathbf{x}'\mathbf{x}}/t_cR)-\alpha^2m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}/t_cR]^2}$$

$$=\frac{m^2_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}[(1+m^1_{\mathbf{x}'\mathbf{x}}/t_cR)-\alpha^2m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}/t_cR]^2}{m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}[(1+m^1_{\mathbf{x}'\mathbf{x}}/t_cR)-\alpha^2m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}/t_cR]^2}=\frac{m^2_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}}{m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}},$$

as desired. Our next task is to show that (B32) is asymptotically strictly less than $m^2_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}/m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}$ if beliefs are not proportional to $\boldsymbol{\beta}_s$.

We begin by noting that asymptotically simple beliefs are given by

$$\text{(B34)}\quad \overline{\boldsymbol{\beta}}_s \xrightarrow{p} \boldsymbol{\beta}_s + c\mathbf{M}^{-1}\boldsymbol{\beta}_s = \boldsymbol{\beta}_s + c\left[\mathbf{V} + \frac{\boldsymbol{\beta}_s\boldsymbol{\beta}'_s}{\boldsymbol{\beta}'\boldsymbol{\beta}}\right]^{-1}\boldsymbol{\beta}_s$$

$$= \boldsymbol{\beta}_s + c\left[\mathbf{V}^{-1} - \frac{\mathbf{V}^{-1}\boldsymbol{\beta}_s\boldsymbol{\beta}'_s\mathbf{V}^{-1}/\boldsymbol{\beta}'\boldsymbol{\beta}}{1+\boldsymbol{\beta}'_s\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta}}\right]\boldsymbol{\beta}_s = \boldsymbol{\beta}_s + \frac{c\mathbf{V}^{-1}\boldsymbol{\beta}_s}{1+\boldsymbol{\beta}'_s\mathbf{V}^{-1}\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta}},$$

where $c = 1 - \boldsymbol{\beta}'_s\boldsymbol{\beta}_s/\boldsymbol{\beta}'\boldsymbol{\beta}$, so using $\mathbf{x} = \overline{\boldsymbol{\beta}}_s\sqrt{R/\overline{\boldsymbol{\beta}}'_s\overline{\boldsymbol{\beta}}_s}$

$$\text{(B35)}\quad m^i_{\boldsymbol{\beta}'_s\mathbf{x}} = \sqrt{\frac{R}{\overline{\boldsymbol{\beta}}'_s\overline{\boldsymbol{\beta}}_s}}\left[m^i_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s} + \frac{cm^{i+1}_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}}{(1+m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}/\boldsymbol{\beta}'\boldsymbol{\beta})}\right]$$

$$\text{and}\quad m^i_{\mathbf{x}'\mathbf{x}} = \frac{R}{\overline{\boldsymbol{\beta}}'_s\overline{\boldsymbol{\beta}}_s}\left[m^i_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s} + \frac{2cm^{i+1}_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}}{(1+m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}/\boldsymbol{\beta}'\boldsymbol{\beta})} + \frac{c^2m^{i+2}_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}}{(1+m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}/\boldsymbol{\beta}'\boldsymbol{\beta})^2}\right].$$

(B35) tells us that all $m^i_{\boldsymbol{\beta}'_s\mathbf{x}}$ and $m^i_{\mathbf{x}'\mathbf{x}}$ can be expressed as a combination of $m^j_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}$ terms. Each $m^j_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}$ is asymptotically bounded, as

$$\text{(B36)}\quad \boldsymbol{\beta}'_s\mathbf{V}^{-j}\boldsymbol{\beta}_s \le \lambda_{\max}(\mathbf{V}^{-j})\boldsymbol{\beta}'_s\boldsymbol{\beta}_s = \frac{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}{\lambda_{\min}(\mathbf{V})^j} \le \frac{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}{\lambda_{\min}\left(\mathbf{V}-\mathbf{X}'_{ss}\mathbf{X}_{ss}/t_cR\right)^j} \le \frac{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}{(\sigma^2_n/R)^j}$$

where we have made use of the definition of $\mathbf{V}$ from (B21). Added to that the fact that (B34) implies that $\overline{\boldsymbol{\beta}}'_s\overline{\boldsymbol{\beta}}_s \ge \boldsymbol{\beta}'_s\boldsymbol{\beta}_s$, and we can see that all $m^i_{\boldsymbol{\beta}'_s\mathbf{x}}$ and $m^i_{\mathbf{x}'\mathbf{x}}$ are bounded from above and the limit of (B32) as t_c goes to infinity is $m^2_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}/m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}$, as should be expected since the rank one updates of $\mathbf{V}$, $\mathbf{x}/(t_cR)^{½}$, get smaller and smaller.

With the preceding in mind, consider (B32) as a function of t_c, $g(t_c)$, with

$$\text{(B37)}\quad g'(t_c) = \left(\boldsymbol{\beta}'_s\left[\mathbf{V}^{-1} - \frac{\mathbf{V}\mathbf{x}^{-1}\mathbf{x}'\mathbf{V}^{-1}/t_cR}{1+\mathbf{x}'\mathbf{V}^{-1}\mathbf{x}/t_cR}\right]\boldsymbol{\beta}_s\right)^{-3}\frac{2m^1_{\boldsymbol{\beta}'_s\mathbf{x}}}{t_c^2R}$$

$$*\left(\underbrace{\frac{m^1_{\boldsymbol{\beta}'_s\mathbf{x}}[m^2_{\boldsymbol{\beta}'_s\mathbf{x}}m^1_{\boldsymbol{\beta}'_s\mathbf{x}} - m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}m^2_{\mathbf{x}'\mathbf{x}}]}{t_cR}}_{c_1} + \underbrace{(1+m^1_{\mathbf{x}'\mathbf{x}}/t_cR)}_{c_2}\underbrace{[m^2_{\boldsymbol{\beta}'_s\mathbf{x}}m^1_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s} - m^2_{\boldsymbol{\beta}'_s\boldsymbol{\beta}_s}m^1_{\boldsymbol{\beta}'_s\mathbf{x}}]}_{c_3}\right).$$

Substituting using (B35), we have

$$\text{(B38)}\quad c_3 = \sqrt{\frac{R}{\bar{\boldsymbol\beta}_s'\bar{\boldsymbol\beta}_s}}\left(\left[m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s} + \frac{c m^3_{\boldsymbol\beta_s'\boldsymbol\beta_s}}{(1+m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}/\boldsymbol\beta'\boldsymbol\beta)}\right] m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s} - m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}\left[m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s} + \frac{c m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}}{(1+m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}/\boldsymbol\beta'\boldsymbol\beta)}\right]\right)$$

$$= \sqrt{\frac{R}{\bar{\boldsymbol\beta}_s'\bar{\boldsymbol\beta}_s}}\frac{c(m^3_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s} - m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s})}{(1+m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}/\boldsymbol\beta'\boldsymbol\beta)} \geq 0,$$

$$\text{as } m^3_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s} = \boldsymbol\beta_s'\mathbf{V}^{-3}\boldsymbol\beta_s\boldsymbol\beta_s'\mathbf{V}^{-1}\boldsymbol\beta_s = (\mathbf{V}^{-3/2}\boldsymbol\beta_s)'(\mathbf{V}^{-3/2}\boldsymbol\beta_s)(\mathbf{V}^{-1/2}\boldsymbol\beta_s)'(\mathbf{V}^{-1/2}\boldsymbol\beta_s)$$

$$\geq (\mathbf{V}^{-3/2}\boldsymbol\beta_s)'(\mathbf{V}^{-1/2}\boldsymbol\beta_s)(\mathbf{V}^{-3/2}\boldsymbol\beta_s)'(\mathbf{V}^{-1/2}\boldsymbol\beta_s) = \boldsymbol\beta_s'\mathbf{V}^{-2}\boldsymbol\beta_s\boldsymbol\beta_s'\mathbf{V}^{-2}\boldsymbol\beta_s = m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s},$$

$$\text{while } c_1 = \frac{m^1_{\boldsymbol\beta_s'\mathbf{x}}}{t_c R}\frac{R}{\bar{\boldsymbol\beta}_s'\bar{\boldsymbol\beta}_s}\left[\frac{c(m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s} - m^3_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s})}{(1+m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}/\boldsymbol\beta'\boldsymbol\beta)} + \frac{c^2(m^3_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s} - m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^4_{\boldsymbol\beta_s'\boldsymbol\beta_s})}{(1+m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}/\boldsymbol\beta'\boldsymbol\beta)^2}\right],$$

where we once again use the Cauchy-Schwarz inequality. We are unable to sign c_1, but since $c_2 > 1$ and $c_3 \geq 0$, if c_1 is strictly positive it follows that $g'(t_c)$ is strictly positive and consequently $g(t_c)$ is strictly less than $m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}/m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}$ for finite t_c as long as simple beliefs are not proportional to $\boldsymbol\beta_s$. Going forward, we assume this is not the case, i.e. that $c_1 \leq 0$.

Using the work above, we formally note the upper bounds on $R/\bar{\boldsymbol\beta}_s'\bar{\boldsymbol\beta}_s$, $m^1_{\boldsymbol\beta_s'\mathbf{x}}$ and the maximum eigenvalue of $\mathbf{V}^{-1}$

$$\text{(B39)}\quad \frac{R}{\boldsymbol\beta_s'\boldsymbol\beta_s} \geq \frac{R}{\bar{\boldsymbol\beta}_s'\bar{\boldsymbol\beta}_s},\ \ m^{1*}_{\boldsymbol\beta_s'\mathbf{x}} = \sqrt{R\boldsymbol\beta_s'\boldsymbol\beta_s}\left[\frac{R}{\sigma_n^2} + c\left(\frac{R}{\sigma_n^2}\right)^2\right] \geq m^1_{\boldsymbol\beta_s'\mathbf{x}},\ \ \&\ \ \lambda^* = \frac{R}{\sigma_n^2} \geq \lambda_{\max}[\mathbf{V}^{-1}],$$

and define t^* as

$$\text{(B40)}\quad t^* = 2\sqrt{\frac{R}{\boldsymbol\beta_s'\boldsymbol\beta_s}}\frac{m^{1*}_{\boldsymbol\beta_s'\mathbf{x}}}{R}\max(1, c\lambda^*).$$

Substituting into $c_1 + c_2c_3$ using (B38) and $t_c > t^*$

$$\text{(B41)}\quad \underbrace{\frac{m^1_{\boldsymbol\beta_s'\mathbf{x}}[m^2_{\boldsymbol\beta_s'\mathbf{x}}m^1_{\boldsymbol\beta_s'\mathbf{x}} - m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^2_{\mathbf{x}'\mathbf{x}}]}{t_c R}}_{\leq \text{ by assumption}} + \underbrace{(1 + m^1_{\mathbf{x}'\mathbf{x}}/t_c R)}_{>1}\underbrace{[m^2_{\boldsymbol\beta_s'\mathbf{x}}m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s} - m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^1_{\boldsymbol\beta_s'\mathbf{x}}]}_{\geq 0 \text{ from (B35)}}$$

$$\geq \frac{m^1_{\boldsymbol\beta_s'\mathbf{x}}}{t^* R}\frac{R}{\bar{\boldsymbol\beta}_s'\bar{\boldsymbol\beta}_s}\left[\frac{c(m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s} - m^3_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s})}{(1+m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}/\boldsymbol\beta'\boldsymbol\beta)} + \frac{c^2(m^3_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s} - m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^4_{\boldsymbol\beta_s'\boldsymbol\beta_s})}{(1+m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}/\boldsymbol\beta'\boldsymbol\beta)^2}\right]$$

$$+ \sqrt{\frac{R}{\bar{\boldsymbol\beta}_s'\bar{\boldsymbol\beta}_s}}\frac{c(m^3_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s} - m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s})}{(1+m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}/\boldsymbol\beta'\boldsymbol\beta)}$$

$$= \underbrace{\sqrt{\frac{R}{\bar{\boldsymbol\beta}_s'\bar{\boldsymbol\beta}_s}}\frac{c(m^3_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}-m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s})}{(1+m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}/\boldsymbol\beta'\boldsymbol\beta)}}_{\geq 0 \text{ from (B38)}}\underbrace{\left[\frac{1}{2}-\sqrt{\frac{R}{\bar{\boldsymbol\beta}_s'\bar{\boldsymbol\beta}_s}}\frac{m^1_{\boldsymbol\beta_s'\mathbf{x}}}{t^*R}\right]}_{\geq 0 \text{ from (B40)}}+$$

$$\frac{m^1_{\boldsymbol\beta_s'\mathbf{x}}c^2}{t^*R(1+m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}/\boldsymbol\beta'\boldsymbol\beta)^2}\frac{R}{\bar{\boldsymbol\beta}_s'\bar{\boldsymbol\beta}_s}\left[(m^3_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}-m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^4_{\boldsymbol\beta_s'\boldsymbol\beta_s})+\left(\begin{array}{c}\underbrace{\sqrt{\frac{\bar{\boldsymbol\beta}_s'\bar{\boldsymbol\beta}_s}{R}}\frac{t^*R(1+m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}/\boldsymbol\beta'\boldsymbol\beta)}{2cm^1_{\boldsymbol\beta_s'\mathbf{x}}}}_{\geq\lambda^* \text{ from (B40)}}\\ *\,(\underbrace{m^3_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}-m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}}_{\geq 0 \text{ from (B38)}})\end{array}\right)\right]$$

$$\geq \frac{m^1_{\boldsymbol\beta_s'\mathbf{x}}c^2}{t^*R(1+m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}/\boldsymbol\beta'\boldsymbol\beta)^2}\frac{R}{\bar{\boldsymbol\beta}_s'\bar{\boldsymbol\beta}_s}\underbrace{\left[m^3_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}-m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^4_{\boldsymbol\beta_s'\boldsymbol\beta_s}+\lambda^*(m^3_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^1_{\boldsymbol\beta_s'\boldsymbol\beta_s}-m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s}m^2_{\boldsymbol\beta_s'\boldsymbol\beta_s})\right]}_{c_4}$$

Focusing on c_4 in the last line, as $m^i_{\boldsymbol\beta_s'\boldsymbol\beta_s}=\boldsymbol\beta_s'\mathbf{V}^{-i}\boldsymbol\beta_s$, we use the spectral decomposition of $\mathbf{V}^{-1}$, as in (B22) earlier

$$\text{(B42)}\quad c_4=\sum_{i=1}^{k_s}\lambda_i^3a_i^2\sum_{i=1}^{k_s}\lambda_i^2a_i^2-\sum_{i=1}^{k_s}\lambda_ia_i^2\sum_{i=1}^{k_s}\lambda_i^4a_i^2+\lambda^*\left[\sum_{i=1}^{k_s}\lambda_i^3a_i^2\sum_{i=1}^{k_s}\lambda_ia_i^2-\sum_{i=1}^{k_s}\lambda_i^2a_i^2\sum_{i=1}^{k_s}\lambda_i^2a_i^2\right]$$

$$=2\sum_{i=1}^{k_s-1}\sum_{j=i+1}^{k_s}(\lambda_i^3\lambda_j^2-\lambda_i\lambda_j^4)a_i^2a_j^2+2\lambda^*\sum_{i=1}^{k_s-1}\sum_{j=i+1}^{k_s}(\lambda_i^3\lambda_j-\lambda_i^2\lambda_j^2)a_i^2a_j^2$$

$$=2\sum_{i=1}^{k_s-1}\sum_{j=i+1}^{k_s}[\lambda_i^3\lambda_j^2-\lambda_i\lambda_j^4+\lambda^*(\lambda_i^3\lambda_j-\lambda_i^2\lambda_j^2)]a_i^2a_j^2$$

$$\geq\sum_{i=1}^{k_s-1}\sum_{j=i+1}^{k_s}[\lambda_i^3\lambda_j^2-\lambda_i\lambda_j^4+\lambda_i(\lambda_i^3\lambda_j-\lambda_i^2\lambda_j^2)]a_i^2a_j^2=\sum_{i=1}^{k_s-1}\sum_{j=i+1}^{k_s}[\lambda_i^4\lambda_j-\lambda_i\lambda_j^4]a_i^2a_j^2\ \geq\ 0,$$

where we have used the fact that the λ_i are ordered in decreasing order, with $\lambda_1 \geq \ldots \geq \lambda_i \ldots \geq \lambda_{k_s}$. The last line of (B42) holds with strict inequality whenever there exists a difference between the maximum and minimum eigenvalues corresponding to non-zero a_i. Strict equality holds when $\lambda_i = \lambda_j = \lambda$ for all $a_i \neq 0$ and $a_j \neq 0$. But in this case, since $\boldsymbol{a} = \mathbf{E}'\boldsymbol\beta_s$, we have $\mathbf{V}^{-1}\boldsymbol\beta_s=\mathbf{E}\boldsymbol\Lambda\mathbf{E}'\boldsymbol\beta_s=\mathbf{E}(\lambda\mathbf{I}_{k_s})\mathbf{a}=\lambda\mathbf{E}\mathbf{a}=\lambda\mathbf{E}\mathbf{E}'\boldsymbol\beta_s=\lambda\boldsymbol\beta_s$, so from (B34) earlier

$$\text{(B43)}\quad \bar{\boldsymbol\beta}_s \xrightarrow{p} \boldsymbol\beta_s+\frac{c\mathbf{V}^{-1}\boldsymbol\beta_s}{1+\boldsymbol\beta_s'\mathbf{V}^{-1}\boldsymbol\beta_s/\boldsymbol\beta'\boldsymbol\beta}=\left[1+\frac{c\lambda}{1+\lambda\boldsymbol\beta_s'\boldsymbol\beta_s/\boldsymbol\beta'\boldsymbol\beta}\right]\boldsymbol\beta_s,$$

that is, simple beliefs are proportional to $\boldsymbol\beta_s$. So, we may assume strict inequality in (B42) and consequently conclude that for all $t_c > t^*$, as long as simple beliefs are not proportional to $\boldsymbol{\beta}_s$, $g'(t_c)$ is strictly positive and hence $g(t_c)$ is strictly less than $m^2_{\boldsymbol{\beta}_s'\boldsymbol{\beta}_s} / m^1_{\boldsymbol{\beta}_s'\boldsymbol{\beta}_s} m^1_{\boldsymbol{\beta}_s'\boldsymbol{\beta}_s}$. This concludes our proof that the rank one update of $\mathbf{X}_{ss}'\mathbf{X}_{ss}$ when the simple are in power lowers the ratio $f(c^2\mathbf{M}^{-2})/f(c\mathbf{M}^{-1})^2$ as long as simple beliefs are not proportional to $\boldsymbol{\beta}_s$, i.e. as long as the economy is not on the (lowest) level curve in Figure B2 associated with the steady state.

To summarize, when the complex are in power, in the formula for $\mathbf{M}$ t_c increases, which increases $f(c\mathbf{M}^{-1})$ and lowers the ratio $f(c^2\mathbf{M}^{-2})/f(c\mathbf{M}^{-1})^2$. When the simple are in power, t_s increases and there is also a rank-one update of $\mathbf{M}$ based upon implemented simple policy. Both of these lower both $f(c\mathbf{M}^{-1})$ and $f(c^2\mathbf{M}^{-2})/f(c\mathbf{M}^{-1})^2$. These are the results stated in (B17a) and (B17b). Turning to (B17c), we begin by noting that since the sum of the eigenvalues of a matrix equals the trace, the individual eigenvalues γ_i of $\mathbf{X}_{ss}'\mathbf{X}_{ss}$ are bounded from above by Rt_s. Consequently, we can bound the derivatives in (B25) and prove that their limit is zero

$$\text{(B44)} \quad 0 < \frac{d\lambda_i}{dt_c} = \frac{R(\gamma_i + t_s\sigma_n^2)}{(\gamma_i + (t_s + t_c)\sigma_n^2)^2} < \frac{R(Rt_s + t_s\sigma_n^2)}{t^2\sigma_n^4} < \frac{R(R+\sigma_n^2)}{t\sigma_n^4}$$

$$\text{and} \quad 0 > \frac{d\lambda_i}{dt_s} = -\frac{Rt_c\sigma_n^2}{(\gamma_i + (t_s+t_c)\sigma_n^2)^2} > -\frac{R}{t\sigma_n^2}$$

$$\Rightarrow \quad 0 \le \lim_{t\to\infty}\frac{d\lambda_i}{dt_c} \le \lim_{t\to\infty}\frac{R(R+\sigma_n^2)}{t\sigma_n^4} = 0 \;\;\&\;\; 0 \ge \lim_{t\to\infty}\frac{d\lambda_i}{dt_s} \ge \lim_{t\to\infty} -\frac{R}{t\sigma_n^2} = 0.$$

The only remaining effect on $f(c\mathbf{M}^{-1})$ with the passage of time is through the rank one update of $\boldsymbol{\beta}_s'\mathbf{V}^{-1}\boldsymbol{\beta}_s$, which, as described earlier in (B31), generates a change

$$\text{(B45)} \quad -\frac{\boldsymbol{\beta}_s'\mathbf{V}^{-1}\mathbf{x}\mathbf{x}'\mathbf{V}^{-1}\boldsymbol{\beta}_s / t_cR}{1+\mathbf{x}'\mathbf{V}^{-1}\mathbf{x}/t_cR} = -\frac{m^1_{\boldsymbol{\beta}_s'\mathbf{x}} m^1_{\boldsymbol{\beta}_s'\mathbf{x}} / t_cR}{1+m^1_{\mathbf{x}'\mathbf{x}}/t_cR}.$$

However, as shown in (B36), all $m^i_{\boldsymbol{\beta}_s'\mathbf{x}}$ and $m^i_{\mathbf{x}'\mathbf{x}}$ are bounded from above, while we established much earlier above that t_c goes to infinity (outside of equilibrium paths of probability measure zero which we are not examining). Consequently, the change in $f(c\mathbf{M}^{-1})$ through this mechanism goes to zero as well. This proves (B17c) and completes the proof of the convergence of $\overline{\boldsymbol{\beta}}_i$ and $\theta_i = t_i/t$ in this appendix.

Appendix II: Proofs for Results on Berk-Nash Equilibria

Proof of Proposition 1: Let the beliefs of type C be degenerate on $\overline{\boldsymbol{\beta}}_c = \boldsymbol{\beta}$ and let the beliefs of type S be degenerate on $\overline{\boldsymbol{\beta}}_s = \tau^* \boldsymbol{\beta}_s$, where $\tau^* = \sqrt{\boldsymbol{\beta}'\boldsymbol{\beta} / \boldsymbol{\beta}_s'\boldsymbol{\beta}_s} > 1$. Let $\mathbf{x}_s^* = \boldsymbol{\beta}_s \sqrt{R / \boldsymbol{\beta}_s'\boldsymbol{\beta}_s}$ and $\mathbf{x}_c^* = \boldsymbol{\beta}\sqrt{R/\boldsymbol{\beta}'\boldsymbol{\beta}}$. We will prove that this configuration, together with some interior value of θ_s, θ_s^*, is a Berk-Nash equilibrium.

First note that given these beliefs and actions, condition (*IV*.1) in the definition of Berk-Nash equilibrium is satisfied.

We now show that there exists $0 < \theta_s^* < 1$ such that conditions (*IV*.2) and (*IV*.3) are satisfied as well. First, we find θ_s^* such that

$$\text{(C1)}\quad \tau^*\boldsymbol{\beta}_s \in \left\{ \arg\min_{\hat{\boldsymbol{\beta}}_s} E_\varepsilon \left[\theta_s^* \ln \frac{f(\varepsilon)}{f(\boldsymbol{\beta}_s'\mathbf{x}_s^* - \hat{\boldsymbol{\beta}}_s'\mathbf{x}_s^* + \varepsilon)} + (1-\theta_s^*) \ln \frac{f(\varepsilon)}{f(\boldsymbol{\beta}'\mathbf{x}_c^* - \hat{\boldsymbol{\beta}}_s'\mathbf{x}_{sc}^* + \varepsilon)} \right] \right\}.$$

We do this in two steps: (i) we will show that given any value for θ_s there is for some τ a $\overline{\boldsymbol{\beta}}_s(\theta_s) = \tau\boldsymbol{\beta}_s$ that is an element of the set of $\hat{\boldsymbol{\beta}}_s$ that minimise the E_ε given in (C1). (ii) we will use (i) and the mean value theorem to show the existence of an interior θ_s^* such that $\tau^*\boldsymbol{\beta}_s$ is an element of the set of $\hat{\boldsymbol{\beta}}_s$ that minimise the E_ε given in (C1).

Proof of substep (i): Note that by the equilibrium configuration that we consider, where $\mathbf{x}_s^* = \tau^*\mathbf{x}_{sc}^*$, we have

$$\text{(C2)}\quad \boldsymbol{\beta}'\mathbf{x}_c^* - \hat{\boldsymbol{\beta}}_s'\mathbf{x}_{sc}^* = (\boldsymbol{\beta}_s - \hat{\boldsymbol{\beta}}_s)' \frac{\mathbf{x}_s^*}{\tau^*} + \boldsymbol{\beta}_{\sim sc}'\mathbf{x}_{\sim sc}^*$$

where *~sc* denotes the policies of type C that are deemed irrelevant by type S. We therefore consider the KL minimizers of

$$\text{(C3)}\ \min_{\hat{\boldsymbol{\beta}}_s} E_\varepsilon \left[\theta_s \ln \frac{f(\varepsilon)}{f((\boldsymbol{\beta}_s - \hat{\boldsymbol{\beta}}_s)'\mathbf{x}_s^* + \varepsilon)} + (1-\theta_s) \ln \frac{f(\varepsilon)}{f((\boldsymbol{\beta}_s - \hat{\boldsymbol{\beta}}_s)'\mathbf{x}_s^*/\tau^* + \boldsymbol{\beta}_{\sim sc}\mathbf{x}_{\sim sc}^* + \varepsilon)} \right].$$

By the assumptions on $f(\varepsilon)$, for any $0 \le \theta_s \le 1$ a solution to the above, i.e. a minimum, exists. Fix $\theta_s \in [0,1]$ and pick such a solution $\hat{\boldsymbol{\beta}}_s(\theta_s)$. This solution satisfies, for some a^* and b^*

$$\text{(C4)}\ (\boldsymbol{\beta}_s - \hat{\boldsymbol{\beta}}_s(\theta_s))'\mathbf{x}_s^* = a^* \quad \text{and} \quad (\boldsymbol{\beta}_s - \hat{\boldsymbol{\beta}}_s(\theta_s))'\mathbf{x}_s^*/\tau^* + \boldsymbol{\beta}_{\sim sc}'\mathbf{x}_{\sim sc}^* = b^*.$$

Plugging the first equality into the second, this system of equations can be written as:

$$\text{(C5)}\ (\boldsymbol{\beta}_s - \hat{\boldsymbol{\beta}}_s(\theta_s))'\mathbf{x}_s^* = a^* \quad \text{and} \quad a^*/\tau^* + \boldsymbol{\beta}_{\sim sc}'\mathbf{x}_{\sim sc}^* = b^*.$$

Note that any solution to these equations will also be a solution to (C3). Therefore any vector $\breve{\boldsymbol{\beta}}_s$ satisfying

$$\text{(C6)}\ (\boldsymbol{\beta}_s - \breve{\boldsymbol{\beta}}_s)'\mathbf{x}_s^* = a^* \quad \text{and} \quad a^*/\tau^* + \boldsymbol{\beta}'_{\sim sc}\mathbf{x}^*_{\sim sc} = b^*$$

is a solution. But the second equation is inconsequent for finding any solution $\breve{\boldsymbol{\beta}}_s$ and merely shows how the colinearity of policies imposes conditions on the values of a^* and b^* at a minimum. Thus, (C6) can be written as:

$$\text{(C7)}\ (\boldsymbol{\beta}_s - \breve{\boldsymbol{\beta}}_s)'\mathbf{x}_s^* = a^*$$

which has multiple solutions, including one in which $\breve{\boldsymbol{\beta}}_s = \tau\boldsymbol{\beta}_s$ for some τ such that

$$\text{(C8)}\ (\boldsymbol{\beta}_s - \tau\boldsymbol{\beta}_s)'\mathbf{x}_s^* = a^*$$

So, without loss of generality, for any $\theta_s \in [0,1]$ there exists a solution to the KL minimisation problem which satisfies $\breve{\boldsymbol{\beta}}_s = \tau\boldsymbol{\beta}_s$, which completes the proof of substep (i).

Proof of substep (ii): We now consider colinear solutions to the KL minimisation problem for different values of θ_s. When $\theta_s = 0$ a colinear solution is achieved where $\boldsymbol{\beta}'\mathbf{x}_c^* - \breve{\boldsymbol{\beta}}'_s\mathbf{x}_{sc}^* = 0$ so that $\tau > \tau^*$. When $\theta_s = 1$ a colinear solution is achieved where $(\boldsymbol{\beta}_s - \breve{\boldsymbol{\beta}}_s)'\mathbf{x}_s^* = 0$ at $\breve{\boldsymbol{\beta}}_s = \boldsymbol{\beta}_s$ so that $\tau = 1 < \tau^*$. By continuity of the minimum value function, there exists $\theta_s^* \in (0,1)$ for which $\breve{\boldsymbol{\beta}}_s = \tau^*\boldsymbol{\beta}_s$ is a solution to (C3). This completes the proof of substep (ii).

The above (i) and (ii) have allowed us to find an interior θ_s^* such that $\tau^*\boldsymbol{\beta}_s$ satisfies (C1), as desired.

We now show that the Berk Nash equilibrium conditions (*IV*.2) and (*IV*.3) are satisfied by the configuration given above. Condition (*IV*.2) is satisified as $\overline{\boldsymbol{\beta}}'_s\overline{\boldsymbol{\beta}}_s = (\tau^*)^2\boldsymbol{\beta}'_s\boldsymbol{\beta}_s = \boldsymbol{\beta}'\boldsymbol{\beta} = \overline{\boldsymbol{\beta}}'_c\overline{\boldsymbol{\beta}}_c$. For condition (*IV*.3) applied to *C*, note that for type *C* the only vector in the support of its belief is $\overline{\boldsymbol{\beta}}_c = \boldsymbol{\beta}$ and

$$\text{(C9)}\ \overline{\boldsymbol{\beta}}_c = \boldsymbol{\beta} \in \left\{ \min_{\hat{\boldsymbol{\beta}}_c} E_\varepsilon\left[\theta_s^* \ln \frac{f(\varepsilon)}{f(\boldsymbol{\beta}'_s\mathbf{x}_s^* - \hat{\boldsymbol{\beta}}'_c\mathbf{x}_{cs}^* + \varepsilon)} + (1-\theta_s^*)\ln \frac{f(\varepsilon)}{f(\boldsymbol{\beta}'_c\mathbf{x}_c^* - \hat{\boldsymbol{\beta}}'_c\mathbf{x}_c^* + \varepsilon)} \right] \right\}.$$

To see this, note by Gibb's inequality the Kullback-Leibler divergence $E_\varepsilon[\ln(f(\varepsilon)/g(\varepsilon))]$ is greater than or equal to zero, with equality if and only if $f(\varepsilon)$ and $g(\varepsilon)$ coincide almost everywhere. As with $\overline{\boldsymbol{\beta}}_c = \boldsymbol{\beta}$ we have $\boldsymbol{\beta}'_s\mathbf{x}_s^* - \overline{\boldsymbol{\beta}}'_c\mathbf{x}_{cs}^* = 0$ and $\boldsymbol{\beta}'_c\mathbf{x}_c^* - \overline{\boldsymbol{\beta}}'_c\mathbf{x}_c^* = 0$ this establishes the claim above.

For condition (*IV*.3) applied to type *S*, by construction we have that $\overline{\boldsymbol{\beta}}_s = \tau^* \boldsymbol{\beta}_s$, the only vector in the support of *S*'s belief, satisfies (C1). This completes the proof that the configuration we started with is a Berk-Nash equilibrium.

Finally, we note that when *f* is normal the first order condition in the minimization of (C1) implies that $\overline{\boldsymbol{\beta}}_s$ is the OLS coefficient and when $\theta_s^* = 1/(1+\tau^*) =$ $\lim_{\sigma_n^2 \to 0}(1-\tau^*\sigma_n^2 / R)/(1+\tau^*)$, as derived in the paper, $\overline{\boldsymbol{\beta}}_s = \tau^* \boldsymbol{\beta}_s$ solves this first order condition.

Proof of Proposition 2: Below, for type $i \in \{S, C\}$, we call a policy an *equilibrium relevant policy* (ERP) if the expected belief of type *i* on the parameter of that policy is non-zero in equilibrium. Note that equilibrium relevant policies are a subset of type *i*'s relevant policies under their subjective model.

It will suffice to show that (i) $\theta_s > 0$ and (ii) *C* having zero expected beliefs on some of their relevant policies, cannot both be violated in a Berk-Nash equilibrium. Assume that they are violated so that $\theta_s = 0$ and the set of ERPs for type *C* includes all relevant policies. This implies that the set of ERPs for type *C* is a strict superset of type *S*'s ERPs. We now show that this will imply that $\theta_s > 0$. Assume to the contrary that $\theta_s = 0$ so that type *C* is in power with probability 1. Condition (*IV*.3) for type *S* will imply that any vector $\hat{\boldsymbol{\beta}}_s$ in the support of their beliefs must minimise

$$\text{(C10)} \quad E_\varepsilon\left[\ln \frac{f(\varepsilon)}{f(\boldsymbol{\beta}'\mathbf{x}_c^* - \hat{\boldsymbol{\beta}}_s'\mathbf{x}_{sc}^* + \varepsilon)}\right].$$

By Gibb's inequality, the Kullback-Leibler divergence is larger than or equal to zero, holding with equality if and only if both densities coincide almost everywhere. Hence, the KL is minimised at $\boldsymbol{\beta}'\mathbf{x}_c^* - \hat{\boldsymbol{\beta}}_s'\mathbf{x}_{sc}^* = 0$ for each $\hat{\boldsymbol{\beta}}_s$ in the support. By linearity, this implies that the mean beliefs of type *S* also satisfy

$$\text{(C11)} \quad \overline{\boldsymbol{\beta}}_s \mathbf{x}_{sc}^* = \boldsymbol{\beta}'\mathbf{x}_c^*.$$

Given that type *C* is in power, its average beliefs similarly satisfy:

$$\text{(C12)} \quad \overline{\boldsymbol{\beta}}_c \mathbf{x}_c^* = \boldsymbol{\beta}'\mathbf{x}_c^*.$$

Note now that *S*'s optimal action given $\overline{\boldsymbol{\beta}}_s$ is $\mathbf{x}_s^*$ rather than $\mathbf{x}_{sc}^*$. Thus:

$$\text{(C13)} \quad \overline{\boldsymbol{\beta}}_c \mathbf{x}_c^* = \boldsymbol{\beta}'\mathbf{x}_c^* = \overline{\boldsymbol{\beta}}_s'\mathbf{x}_{sc}^* < \overline{\boldsymbol{\beta}}_s'\mathbf{x}_s^*.$$

Noting that $\mathbf{x}_j^* = \overline{\boldsymbol{\beta}}_j \sqrt{R / \overline{\boldsymbol{\beta}}_j' \overline{\boldsymbol{\beta}}_j}$, and hence $\overline{\boldsymbol{\beta}}_j' \mathbf{x}_j^* = \sqrt{R} \sqrt{\overline{\boldsymbol{\beta}}_j' \overline{\boldsymbol{\beta}}_j}$ for $j \in \{S, C\}$, we have:

$$\text{(C14)} \quad \sqrt{R}\sqrt{\overline{\boldsymbol{\beta}}_c' \overline{\boldsymbol{\beta}}_c} = \overline{\boldsymbol{\beta}}_c' \mathbf{x}_c^* = \overline{\boldsymbol{\beta}}_s' \mathbf{x}_{sc}^* < \overline{\boldsymbol{\beta}}_s' \mathbf{x}_s^* = \sqrt{R}\sqrt{\overline{\boldsymbol{\beta}}_s' \overline{\boldsymbol{\beta}}_s} \;\Rightarrow\; \sqrt{\overline{\boldsymbol{\beta}}_c' \overline{\boldsymbol{\beta}}_c} < \sqrt{\overline{\boldsymbol{\beta}}_s' \overline{\boldsymbol{\beta}}_s}.$$

Therefore by equilibrium condition (*IV*.2), $\theta_s = 1$, in contradiction to our initial assumption that $\theta_s = 0$.

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